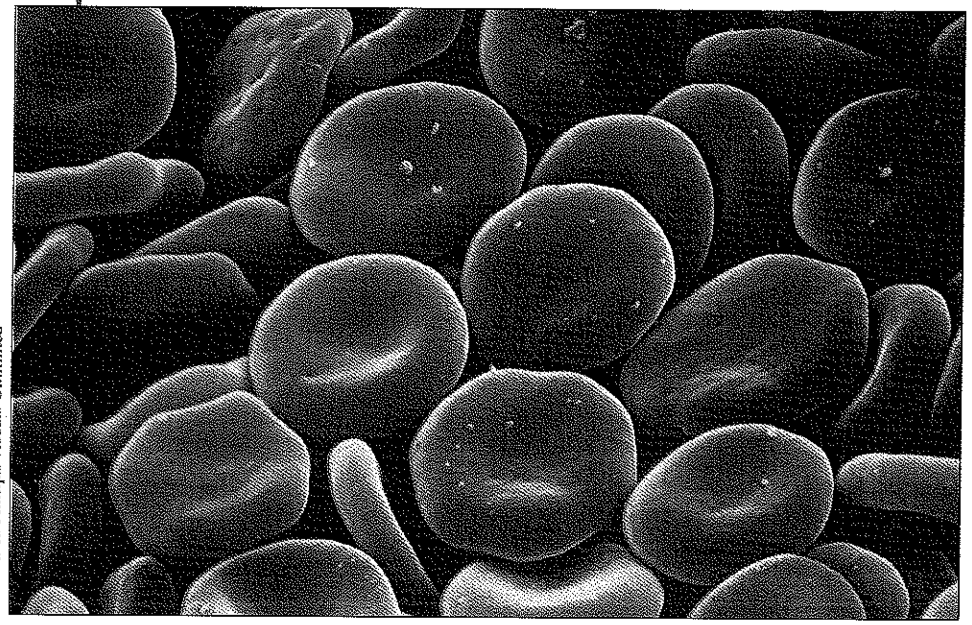
Trigonometric functions are used in medicine to calculate the pressure against the walls of blood vessels. One cycle of the trigonometric model corresponds to one heartbeat.



David Phillips/Visuals Unlimited

4 Trigonometric Functions

What You Should Learn

In this chapter, you will learn how to:

- Describe an angle and convert between degree and radian measure.
- Identify a unit circle and describe its relationship to real numbers.
- Evaluate trigonometric functions of any angle.
- Use fundamental trigonometric identities.
- Sketch graphs of trigonometric functions.
- Evaluate inverse trigonometric functions.
- Evaluate the compositions of trigonometric functions.
- Use trigonometric functions to model and solve real-life problems.

4.1 Radian and Degree Measure

Angles

As derived from the Greek language, the word **trigonometry** means "measurement of triangles." Initially, trigonometry dealt with relationships among the sides and angles of triangles and was used in the development of astronomy, navigation, and surveying. With the development of calculus and the physical sciences in the 17th century, a different perspective arose—one that viewed the classic trigonometric relationships as *functions* with the set of real numbers as their domains. Consequently, the applications of trigonometry expanded to include a vast number of physical phenomena involving rotations and vibrations, including sound waves, light rays, planetary orbits, vibrating strings, pendulums, and orbits of atomic particles.

The approach in this text incorporates *both* perspectives, starting with angles and their measure.

What you should learn

- Describe angles.
- Use radian measure.
- Use degree measure and convert between degree and radian measure.
- Use angles to model and solve real-life problems.

Why you should learn it

Radian measures of angles are involved in numerous aspects of our daily lives. For instance, in Exercise 95 on page 258 you are asked to determine measures of angles of figure skating jumps.


Bob Martin/Getty Images

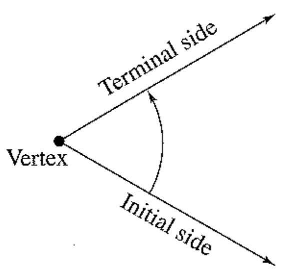


Figure 4.1

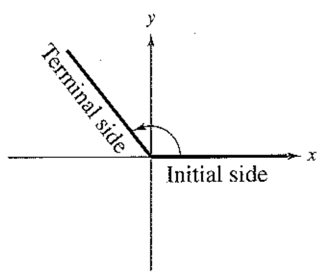


Figure 4.2

An **angle** is determined by rotating a ray (half-line) about its endpoint. The starting position of the ray is the **initial side** of the angle, and the position after rotation is the **terminal side,** as shown in Figure 4.1. The endpoint of the ray is the **vertex** of the angle. This perception of an angle fits a coordinate system in which the origin is the vertex and the initial side coincides with the positive x-axis. Such an angle is in **standard position,** as shown in Figure 4.2. **Positive angles** are generated by counterclockwise rotation, and **negative angles** by clockwise rotation, as shown in Figure 4.3. Angles are labeled with Greek letters such as α (alpha), β (beta), and θ (theta), as well as uppercase letters such as A, B, and C. In Figure 4.4, note that angles α and β have the same initial and terminal sides. Such angles are **coterminal.**

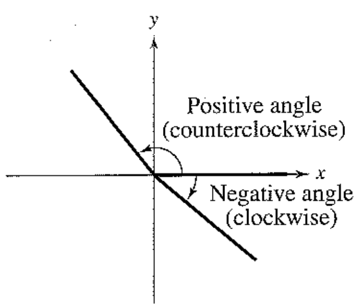


Figure 4.3

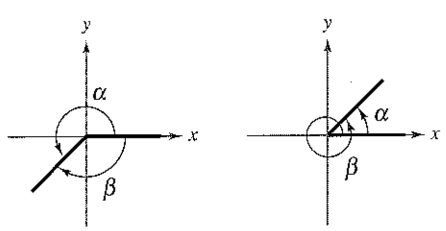
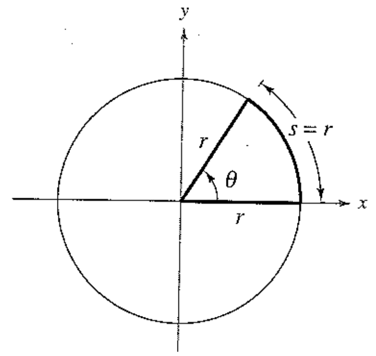


Figure 4.4

Radian Measure

The measure of an angle is determined by the amount of rotation from the initial side to the terminal side. One way to measure angles is in radians. This type of measure is especially useful in calculus. To define a radian, you can use a **central angle** of a circle, one whose vertex is the center of the circle, as shown in Figure 4.5.



Arc length = radius when $\theta = 1$ radian.
Figure 4.5

> **Definition of Radian**
>
> One **radian** is the measure of a central angle θ that intercepts an arc s equal in length to the radius r of the circle. See Figure 4.5.

Because the circumference of a circle is $2\pi r$ units, it follows that a central angle of one full revolution (counterclockwise) corresponds to an arc length of $s = 2\pi r$. Therefore, 2π radians corresponds to 360°, π radians corresponds to 180°, and $\pi/2$ radians corresponds to 90°. Moreover, because $2\pi \approx 6.28$, there are just over six radius lengths in a full circle, as shown in Figure 4.6. In general, the radian measure of a central angle θ is obtained by dividing the arc length s by r. That is, $s/r = \theta$, where θ is *measured in radians*. Because the units of measure for s and r are the same, this ratio has no units—it is simply a real number.

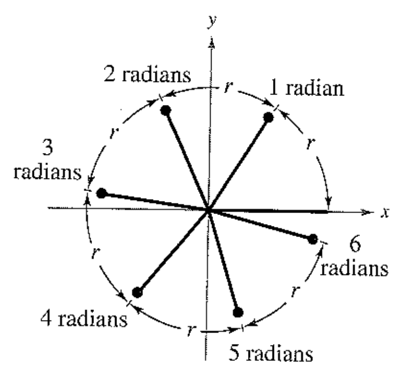


Figure 4.6

Because the radian measure of an angle of one full revolution is 2π, you can obtain the following.

$$\frac{1}{2}\text{ revolution} = \frac{2\pi}{2} = \pi \text{ radians}$$

$$\frac{1}{4}\text{ revolution} = \frac{2\pi}{4} = \frac{\pi}{2} \text{ radians}$$

$$\frac{1}{6}\text{ revolution} = \frac{2\pi}{6} = \frac{\pi}{3} \text{ radians}$$

These and other common angles are shown in Figure 4.7.

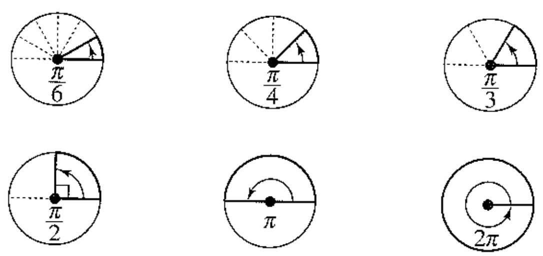


Figure 4.7

Recall that the four quadrants in a coordinate system are numbered I, II, III, and IV. Figure 4.8 shows which angles between 0 and 2π lie in each of the four quadrants. Note that angles between 0 and $\pi/2$ are **acute** and that angles between $\pi/2$ and π are **obtuse.**

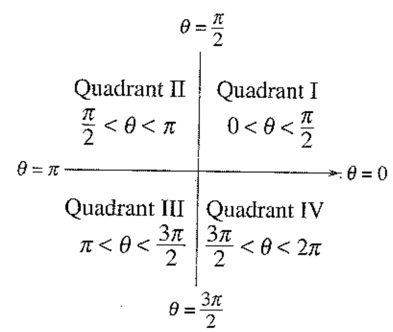


Figure 4.8

Two angles are coterminal if they have the same initial and terminal sides. For instance, the angles 0 and 2π are coterminal, as are the angles $\pi/6$ and $13\pi/6$. You can find an angle that is coterminal to a given angle θ by adding or subtracting 2π (one revolution), as demonstrated in Example 1. A given angle θ has infinitely many coterminal angles. For instance, $\theta = \pi/6$ is coterminal with

$$\frac{\pi}{6} + 2n\pi, \text{ where } n \text{ is an integer.}$$

Example 1 Sketching and Finding Coterminal Angles

a. For the positive angle $\theta = \dfrac{13\pi}{6}$, subtract 2π to obtain a coterminal angle

$$\frac{13\pi}{6} - 2\pi = \frac{\pi}{6}.$$ See Figure 4.9.

b. For the positive angle $\theta = \dfrac{3\pi}{4}$, subtract 2π to obtain a coterminal angle

$$\frac{3\pi}{4} - 2\pi = -\frac{5\pi}{4}.$$ See Figure 4.10.

c. For the negative angle $\theta = -\dfrac{2\pi}{3}$, add 2π to obtain a coterminal angle

$$-\frac{2\pi}{3} + 2\pi = \frac{4\pi}{3}.$$ See Figure 4.11.

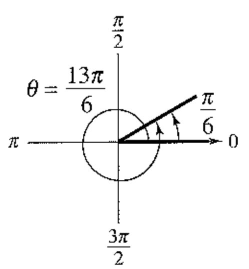


Figure 4.9

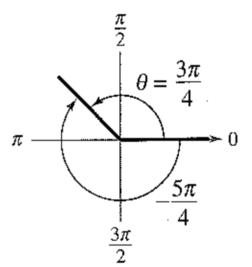


Figure 4.10

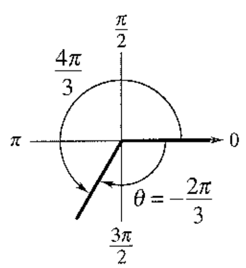


Figure 4.11

✓ ***Checkpoint*** Now try Exercise 13.

Two positive angles α and β are **complementary** (complements of each other) if their sum is $\pi/2$. Two positive angles are **supplementary** (supplements of each other) if their sum is π. See Figure 4.12.

Complementary angles   Supplementary angles
Figure 4.12

STUDY TIP

The phrase "the terminal side of θ lies in a quadrant" is often abbreviated by simply saying that "θ lies in a quadrant." The terminal sides of the "quadrant angles" 0, $\pi/2$, π, and $3\pi/2$ do not lie within quadrants.

Example 2 Complementary and Supplementary Angles

If possible, find the complement and the supplement of (a) $\frac{2\pi}{5}$ and (b) $\frac{4\pi}{5}$.

Solution

a. The complement of $\frac{2\pi}{5}$ is

$$\frac{\pi}{2} - \frac{2\pi}{5} = \frac{5\pi}{10} - \frac{4\pi}{10}$$

$$= \frac{\pi}{10}.$$

The supplement of $\frac{2\pi}{5}$ is

$$\pi - \frac{2\pi}{5} = \frac{5\pi}{5} - \frac{2\pi}{5}$$

$$= \frac{3\pi}{5}.$$

b. Because $4\pi/5$ is greater than $\pi/2$, it has no complement. (Remember that complements are *positive* angles.) The supplement is

$$\pi - \frac{4\pi}{5} = \frac{5\pi}{5} - \frac{4\pi}{5}$$

$$= \frac{\pi}{5}.$$

Checkpoint Now try Exercise 17.

Degree Measure

A second way to measure angles is in terms of **degrees,** denoted by the symbol °. A measure of one degree (1°) is equivalent to a rotation of $\frac{1}{360}$ of a complete revolution about the vertex. To measure angles, it is convenient to mark degrees on the circumference of a circle, as shown in Figure 4.13. So, a full revolution (counterclockwise) corresponds to 360°, a half revolution to 180°, a quarter revolution to 90°, and so on.

Because 2π radians corresponds to one complete revolution, degrees and radians are related by the equations

$$360° = 2\pi \text{ rad} \qquad \text{and} \qquad 180° = \pi \text{ rad}.$$

From the second equation, you obtain

$$1° = \frac{\pi}{180} \text{ rad} \qquad \text{and} \qquad 1 \text{ rad} = \frac{180°}{\pi}$$

which lead to the conversion rules at the top of the next page.

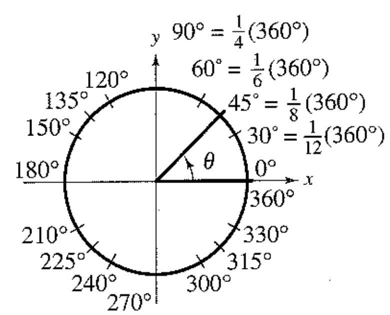


Figure 4.13

Conversions Between Degrees and Radians

1. To convert degrees to radians, multiply degrees by $\frac{\pi \text{ rad}}{180°}$.

2. To convert radians to degrees, multiply radians by $\frac{180°}{\pi \text{ rad}}$.

To apply these two conversion rules, use the basic relationship π rad $= 180°$. (See Figure 4.14.)

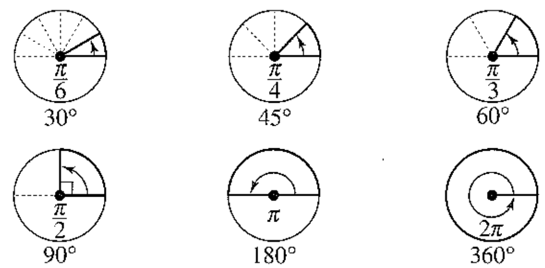


Figure 4.14

When no units of angle measure are specified, *radian measure is implied.* For instance, if you write $\theta = \pi$ or $\theta = 2$, you imply that $\theta = \pi$ radians or $\theta = 2$ radians.

Example 3 Converting from Degrees to Radians

a. $135° = (135 \cancel{\text{deg}})\left(\frac{\pi \text{ rad}}{180 \cancel{\text{deg}}}\right) = \frac{3\pi}{4}$ radians Multiply by $\frac{\pi}{180}$.

b. $540° = (540 \cancel{\text{deg}})\left(\frac{\pi \text{ rad}}{180 \cancel{\text{deg}}}\right) = 3\pi$ radians Multiply by $\frac{\pi}{180}$.

c. $-270° = (-270 \cancel{\text{deg}})\left(\frac{\pi \text{ rad}}{180 \cancel{\text{deg}}}\right) = -\frac{3\pi}{2}$ radians Multiply by $\frac{\pi}{180}$.

✓ ***Checkpoint*** Now try Exercise 45.

Example 4 Converting from Radians to Degrees

a. $-\frac{\pi}{2} \text{ rad} = \left(-\frac{\pi}{2} \cancel{\text{rad}}\right)\left(\frac{180 \text{ deg}}{\pi \cancel{\text{rad}}}\right) = -90°$ Multiply by $\frac{180}{\pi}$.

b. $2 \text{ rad} = (2 \cancel{\text{rad}})\left(\frac{180 \text{ deg}}{\pi \cancel{\text{rad}}}\right) = \frac{360}{\pi} \approx 114.59°$ Multiply by $\frac{180}{\pi}$.

c. $\frac{9\pi}{2} \text{ rad} = \left(\frac{9\pi}{2} \cancel{\text{rad}}\right)\left(\frac{180 \text{ deg}}{\pi \cancel{\text{rad}}}\right) = 810°$ Multiply by $\frac{180}{\pi}$.

✓ ***Checkpoint*** Now try Exercise 51.

TECHNOLOGY TIP

With calculators it is convenient to use *decimal* degrees to denote fractional parts of degrees. Historically, however, fractional parts of degrees were expressed in *minutes* and *seconds*, using the prime ($'$) and double prime ($''$) notations, respectively. That is,

$$1' = \text{one minute} = \tfrac{1}{60}(1°)$$

$$1'' = \text{one second} = \tfrac{1}{3600}(1°).$$

Consequently, an angle of 64 degrees, 32 minutes, and 47 seconds was represented by $\theta = 64° \, 32' \, 47''$.

Many calculators have special keys for converting angles in degrees, minutes, and seconds (D° M′ S″) into decimal degree form, and vice versa.

Linear and Angular Speed

The *radian measure* formula $\theta = s/r$ can be used to measure arc length along a circle. Specifically, for a circle of radius r, a central angle θ (θ is measured in radians) intercepts an arc of length s given by

$s = r\theta.$ Length of circular arc

Example 5 Finding Arc Length

A circle has a radius of 4 inches. Find the length of the arc intercepted by a central angle of 240°, as shown in Figure 4.15.

Solution

To use the formula $s = r\theta$, first convert 240° to radian measure.

$$240° = (240 \text{ deg})\left(\frac{\pi \text{ rad}}{180 \text{ deg}}\right) = \frac{4\pi}{3} \text{ radians}$$

Then, using a radius of $r = 4$ inches, you can find the arc length to be

$$s = r\theta \quad \text{Length of circular arc}$$

$$= 4\left(\frac{4\pi}{3}\right) \quad \text{Substitute for } r \text{ and } \theta.$$

$$= \frac{16\pi}{3} \approx 16.76 \text{ inches.} \quad \text{Simplify.}$$

Note that the units for $r\theta$ are determined by the units for r because θ is given in radian measure and therefore has no units.

✓ ***Checkpoint*** Now try Exercise 85.

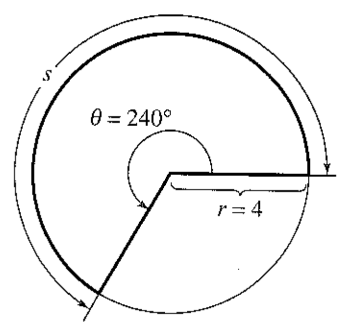


Figure 4.15

The formula for the length of a circular arc can be used to analyze the motion of a particle moving at a *constant speed* along a circular path.

> **Linear and Angular Speed**
>
> Consider a particle moving at a constant speed along a circular arc of radius r. If s is the length of the arc traveled in time t, then the **linear speed** of the particle is
>
> $$\text{Linear speed} = \frac{\text{arc length}}{\text{time}} = \frac{s}{t}.$$
>
> Moreover, if θ is the angle (in radian measure) corresponding to the arc length s, then the **angular speed** of the particle is
>
> $$\text{Angular speed} = \frac{\text{central angle}}{\text{time}} = \frac{\theta}{t}.$$
>
> Linear speed measures how fast the particle moves, and angular speed measures how fast the angle changes.

Example 6 Finding Linear Speed

The second hand of a clock is 10.2 centimeters long, as shown in Figure 4.16. Find the linear speed of the tip of this second hand.

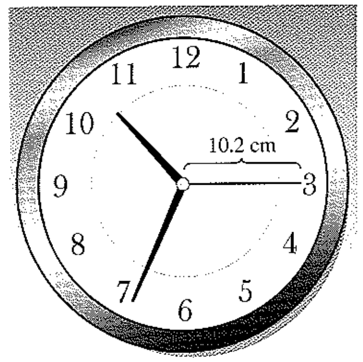


Figure 4.16

Solution

In one revolution, the arc length traveled is

$$s = 2\pi r$$

$$= 2\pi(10.2) \qquad \text{Substitute for } r.$$

$$= 20.4\pi \text{ centimeters.}$$

The time required for the second hand to travel this distance is

$$t = 1 \text{ minute} = 60 \text{ seconds.}$$

So, the linear speed of the tip of the second hand is

$$\text{Linear speed} = \frac{s}{t}$$

$$= \frac{20.4\pi \text{ centimeters}}{60 \text{ seconds}} \approx 1.068 \text{ centimeters per second.}$$

✓ *Checkpoint* Now try Exercise 96.

Example 7 Finding Angular and Linear Speed

A lawn roller with a 10-inch radius makes 1.2 revolutions per second (see Figure 4.17).

a. Find the angular speed of the roller in radians per second.

b. Find the speed of the tractor that is pulling the roller.

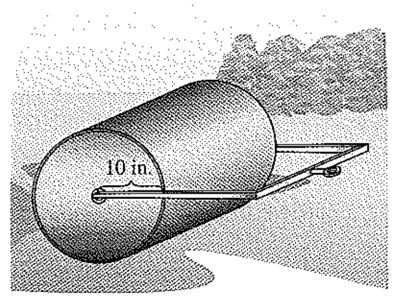


Figure 4.17

Solution

a. Because each revolution generates 2π radians, it follows that the roller turns $(1.2)(2\pi) = 2.4\pi$ radians per second. In other words, the angular speed is

$$\text{Angular speed} = \frac{\theta}{t}$$

$$= \frac{2.4\pi \text{ radians}}{1 \text{ second}} = 2.4\pi \text{ radians per second.}$$

b. The linear speed is

$$\text{Linear speed} = \frac{s}{t} = \frac{r\theta}{t}$$

$$= \frac{10(2.4\pi) \text{ inches}}{1 \text{ second}} \approx 75.4 \text{ inches per second.}$$

✓ *Checkpoint* Now try Exercise 97.

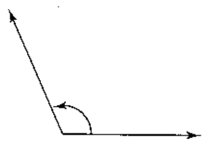
4.1 Exercises

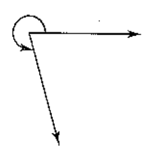
Vocabulary Check

Fill in the blanks.

1. _______ means "measurement of triangles."
2. An _______ is determined by rotating a ray about its endpoint.
3. An angle with its initial side coinciding with the positive x-axis and the origin as its vertex is said to be in _______ .
4. Two angles that have the same initial and terminal sides are _______ .
5. One _______ is the measure of a central angle that intercepts an arc equal in length to the radius of the circle.
6. Two positive angles that have a sum of $\pi/2$ are _______ angles.
7. Two positive angles that have a sum of π are _______ angles.
8. The angle measure that is equivalent to $\frac{1}{360}$ of a complete revolution about an angle's vertex is one _______ .
9. The _______ speed of a particle is the ratio of the arc length traveled to the time traveled.
10. The _______ speed of a particle is the ratio of the change in the central angle to the time.

In Exercises 1–4, estimate the angle to the nearest one-half radian.

1. 

2. 

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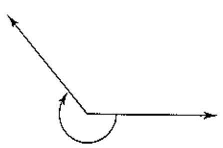
4. 

In Exercises 5–8, determine the quadrant in which each angle lies. (The angle is given in radians.)

5. (a) $\frac{\pi}{5}$ (b) $\frac{7\pi}{5}$

6. (a) $-\frac{\pi}{12}$ (b) $-\frac{11\pi}{9}$

7. (a) -1 (b) -2

8. (a) 3.5 (b) 2.25

In Exercises 9–12, sketch each angle in standard position.

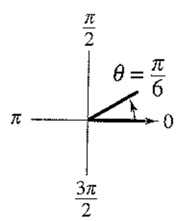
9. (a) $\frac{3\pi}{4}$ (b) $\frac{4\pi}{3}$

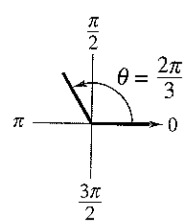
10. (a) $-\frac{7\pi}{4}$ (b) $-\frac{5\pi}{2}$

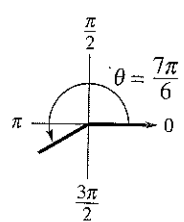
11. (a) $\frac{11\pi}{6}$ (b) $\frac{2\pi}{3}$

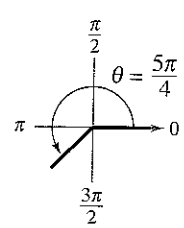
12. (a) 4 (b) -3

In Exercises 13–16, determine two coterminal angles in radian measure (one positive and one negative) for each angle.

13. (a) 

(b) 

14. (a) 

(b) 

15. (a) $-\frac{7\pi}{4}$ (b) $-\frac{11\pi}{6}$

16. (a) $\frac{3\pi}{4}$ (b) $\frac{5\pi}{6}$

In Exercises 17–22, find (if possible) the complement and supplement of the angle.

17. $\frac{\pi}{3}$

18. $\frac{3\pi}{4}$

19. $\frac{\pi}{12}$

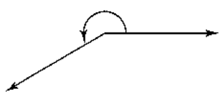
20. $\frac{11\pi}{12}$

21. 1

22. 2

In Exercises 23–26, estimate the number of degrees in the angle.

23.



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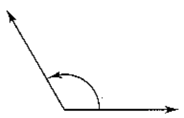

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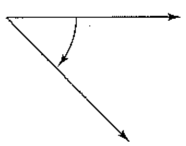

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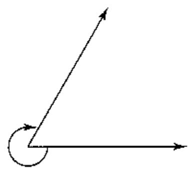


In Exercises 27–30, determine the quadrant in which each angle lies.

27. (a) 150° (b) 282°

28. (a) 7.9° (b) 257.5°

29. (a) $-132° 50'$ (b) $-336° 30'$

30. (a) $-260.25°$ (b) $-2.4°$

In Exercises 31–34, sketch each angle in standard position.

31. (a) 30° (b) 150°

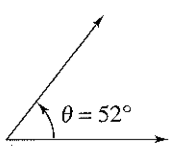
32. (a) $-270°$ (b) $-120°$

33. (a) 405° (b) 780°

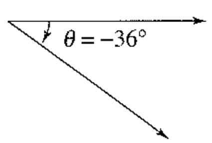
34. (a) $-450°$ (b) $-600°$

In Exercises 35–38, determine two coterminal angles in degree measure (one positive and one negative) for each angle.

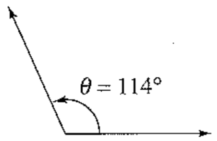
35. (a)



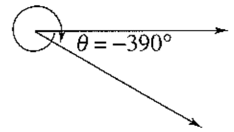
(b)



36. (a)



(b)



37. (a) 300° (b) 230°

38. (a) $-445°$ (b) $-740°$

In Exercises 39–44, find (if possible) the complement and supplement of the angle.

39. 36°

40. 109°

41. 158°

42. 61°

43. 99°

44. 75°

In Exercises 45–48, rewrite each angle in radian measure as a multiple of π. (Do not use a calculator.)

45. (a) 30° (b) 150°

46. (a) 315° (b) 120°

47. (a) $-20°$ (b) $-240°$

48. (a) $-270°$ (b) 144°

In Exercises 49–52, rewrite each angle in degree measure. (Do not use a calculator.)

49. (a) $\frac{3\pi}{2}$ (b) $-\frac{7\pi}{6}$

50. (a) -4π (b) 3π

51. (a) $\frac{7\pi}{3}$ (b) $-\frac{13\pi}{60}$

52. (a) $-\frac{15\pi}{6}$ (b) $\frac{28\pi}{15}$

In Exercises 53–58, convert the angle measure from degrees to radians. Round your answer to three decimal places.

53. 115°

54. 83.7°

55. $-216.35°$

56. $-46.52°$

57. $-0.78°$

58. 395°

In Exercises 59–64, convert the angle measure from radians to degrees. Round your answer to three decimal places.

59. $\frac{\pi}{7}$

60. $\frac{8\pi}{13}$

61. 6.5π

62. -4.2π

63. -2

64. -0.48

In Exercises 65–70, use the angle-conversion capabilities of a graphing utility to convert the angle measure to decimal degree form. Round your answer to three decimal places if necessary.

65. $64° 45'$ **66.** $-124° 30'$

67. $85° 18' 30''$ **68.** $-408° 16' 25''$

69. $-125° 36''$ **70.** $330° 25''$

In Exercises 71–76, use the angle-conversion capabilities of a graphing utility to convert the angle measure to D° M′ S″ form.

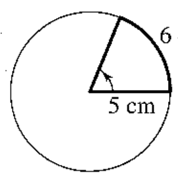
71. $280.6°$ **72.** $-115.8°$

73. $-345.12°$ **74.** $310.75°$

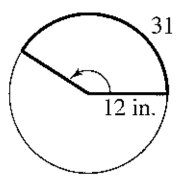
75. -0.355 **76.** 0.7865

In Exercises 77–80, find the angle in radians.

77.



78.



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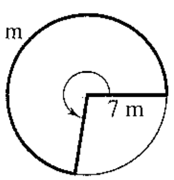
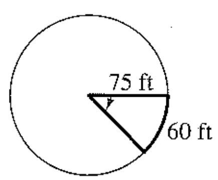


80.



In Exercises 81–84, find the radian measure of the central angle of a circle of radius *r* that intercepts an arc of length *s*.

	Radius r	*Arc Length s*
81.	15 inches	8 inches
82.	22 feet	10 feet
83.	14.5 centimeters	35 centimeters
84.	80 kilometers	160 kilometers

In Exercises 85–88, find the length of the arc on a circle of radius *r* intercepted by a central angle θ.

	Radius r	*Central Angle* θ
85.	14 inches	$180°$
86.	9 feet	$60°$
87.	27 meters	$\frac{2\pi}{3}$ radians
88.	12 centimeters	$\frac{3\pi}{4}$ radians

Distance **In Exercises 89 and 90, find the distance between the cities. Assume that Earth is a sphere of radius 4000 miles and the cities are on the same longitude (one city is due north of the other).**

	City	*Latitude*
89.	Miami	$25° 46' 26''$N
	Erie	$42° 7' 45''$N
90.	Johannesburg, South Africa	$26° 11'$S
	Jerusalem, Israel	$31° 47'$N

91. ***Difference in Latitudes*** Assuming that Earth is a sphere of radius 6378 kilometers, what is the difference in the latitudes of Syracuse, New York and Annapolis, Maryland, where Syracuse is 450 kilometers due north of Annapolis?

92. ***Difference in Latitudes*** Assuming that Earth is a sphere of radius 6378 kilometers, what is the difference in the latitudes of Lynchburg, Virginia and Myrtle Beach, South Carolina, where Lynchburg is 400 kilometers due north of Myrtle Beach?

93. ***Instrumentation*** A voltmeter's pointer is 6 centimeters in length (see figure). Find the angle through which it rotates when it moves 2.5 centimeters on the scale.

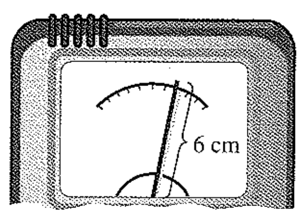


Figure for 93

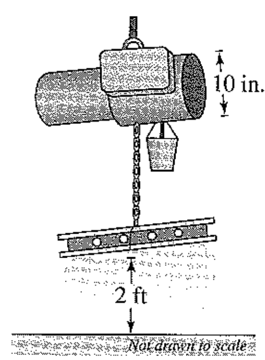


Figure for 94

94. ***Electric Hoist*** An electric hoist is used to lift a piece of equipment 2 feet (see figure). The diameter of the drum on the hoist is 10 inches. Find the number of degrees through which the drum must rotate.

95. *Sports* The number of revolutions made by a figure skater for each type of axel jump is given. Determine the measure of the angle generated as the skater performs each jump. Give the answer in both degrees and radians.

(a) Single axel: $1\frac{1}{2}$ (b) Double axel: $2\frac{1}{2}$

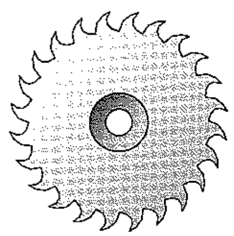
(c) Triple axel: $3\frac{1}{2}$

96. *Linear Speed* A satellite in circular orbit 1250 kilometers above Earth makes one complete revolution every 110 minutes. What is its linear speed? Assume that Earth is a sphere of radius 6400 kilometers.

97. *Construction* The circular blade on a saw has a diameter of 7.5 inches and rotates at 2400 revolutions per minute (see figure).

(a) Find the angular speed in radians per second.

(b) Find the linear speed of the saw teeth (in feet per second) as they contact the wood being cut.



98. *Angular Speed* A car is moving at a rate of 40 miles per hour, and the diameter of its wheels is 2.5 feet.

(a) Find the number of revolutions per minute the wheels are rotating.

(b) Find the angular speed of the wheels in radians per minute.

Synthesis

***True or False?* In Exercises 99–101, determine whether the statement is true or false. Justify your answer.**

99. A degree is a larger unit of measure than a radian.

100. An angle that measures $-1260°$ lies in Quadrant III.

101. The angles of a triangle can have radian measures $\frac{2\pi}{3}$, $\frac{\pi}{4}$, and $\frac{\pi}{12}$.

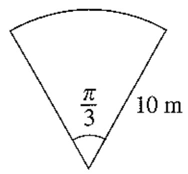
102. *Writing* In your own words, explain the meanings of (a) an angle in standard position, (b) a negative angle, (c) coterminal angles, and (d) an obtuse angle.

103. *Writing* If the radius of a circle is increasing and the magnitude of a central angle is held constant, how is the length of the intercepted arc changing? Explain your reasoning.

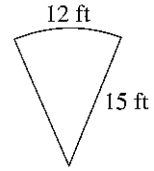
104. *Geometry* Show that the area of a circular sector of radius r with central angle θ is $A = \frac{1}{2}r^2\theta$, where θ is measured in radians.

***Geometry* In Exercises 105 and 106, use the result of Exercise 104 to find the area of the sector.**

105.



106.



107. *Graphical Reasoning* The formulas for the area of a circular sector and arc length are $A = \frac{1}{2}r^2\theta$ and $s = r\theta$, respectively. (r is the radius and θ is the angle measured in radians.)

(a) If $\theta = 0.8$, write the area and arc length as functions of r. What is the domain of each function? Use a graphing utility to graph the functions. Use the graphs to determine which function changes more rapidly as r increases. Explain.

(b) If $r = 10$ centimeters, write the area and arc length as functions of θ. What is the domain of each function? Use a graphing utility to graph and identify the functions.

108. *Writing* A fan motor turns at a given angular speed. How does the speed of the tips of the blades change if a fan of greater diameter is installed on the motor? Explain.

Review

In Exercises 109–114, sketch the graph of $y = x^5$ and the specified transformations.

109. $f(x) = (x - 2)^5$

110. $f(x) = x^5 - 4$

111. $f(x) = 2 - x^5$

112. $f(x) = -(x + 3)^5$

113. $f(x) = (x + 1)^5 - 3$

114. $f(x) = (x - 5)^5 + 1$

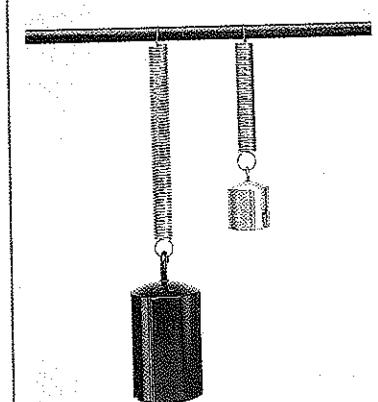
4.2 Trigonometric Functions: The Unit Circle

What you should learn

- Identify a unit circle and describe its relationship to real numbers.
- Evaluate trigonometric functions using the unit circle.
- Use domain and period to evaluate sine and cosine functions.
- Use a calculator to evaluate trigonometric functions.

Why you should learn it

Trigonometric functions are used to model the movement of an oscillating weight. For instance, in Exercise 59 on page 265, the displacement from equilibrium of an oscillating weight suspended by a spring is modeled as a function of time.



Richard Megna/Fundamental Photographs

The Unit Circle

The two historical perspectives of trigonometry incorporate different methods for introducing the trigonometric functions. Our first introduction to these functions is based on the unit circle.

Consider the **unit circle** given by

$$x^2 + y^2 = 1 \qquad \text{Unit circle}$$

as shown in Figure 4.18.

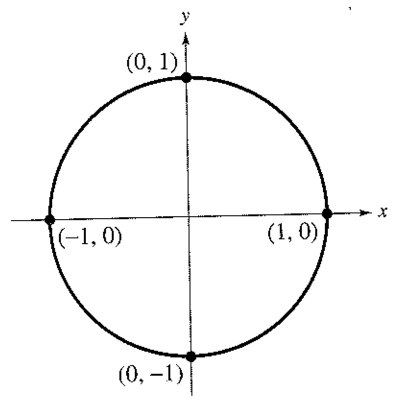
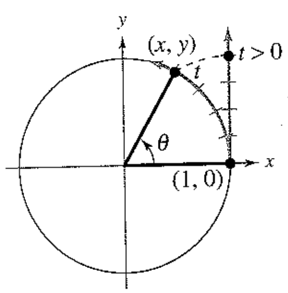


Figure 4.18

Imagine that the real number line is wrapped around this circle, with positive numbers corresponding to a counterclockwise wrapping and negative numbers corresponding to a clockwise wrapping, as shown in Figure 4.19.



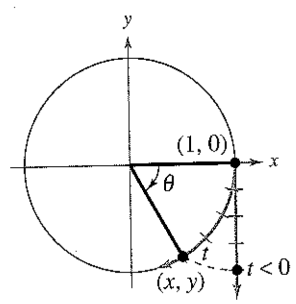


Figure 4.19

As the real number line is wrapped around the unit circle, each real number t corresponds to a point (x, y) on the circle. For example, the real number 0 corresponds to the point $(1, 0)$. Moreover, because the unit circle has a circumference of 2π, the real number 2π also corresponds to the point $(1, 0)$.

In general, each real number t also corresponds to a central angle θ (in standard position) whose radian measure is t. With this interpretation of t, the arc length formula $s = r\theta$ (with $r = 1$) indicates that the real number t is the length of the arc intercepted by the angle θ, given in radians.

The Trigonometric Functions

From the preceding discussion, it follows that the coordinates x and y are two functions of the real variable t. You can use these coordinates to define the six trigonometric functions of t.

sine	**cosecant**
cosine	**secant**
tangent	**cotangent**

These six functions are normally abbreviated sin, csc, cos, sec, tan, and cot, respectively.

Definition of Trigonometric Functions

Let t be a real number and let (x, y) be the point on the unit circle corresponding to t.

$\sin t = y \qquad \csc t = \frac{1}{y}, \quad y \neq 0$

$\cos t = x \qquad \sec t = \frac{1}{x}, \quad x \neq 0$

$\tan t = \frac{y}{x}, \quad x \neq 0 \qquad \cot t = \frac{x}{y}, \quad y \neq 0$

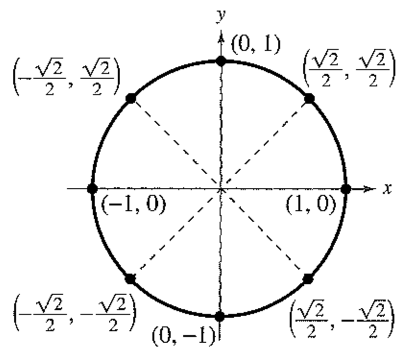


Figure 4.20

Note that the functions in the second column are the *reciprocals* of the corresponding functions in the first column.

In the definitions of the trigonometric functions, note that the tangent and secant are not defined when $x = 0$. For instance, because $t = \pi/2$ corresponds to $(x, y) = (0, 1)$, it follows that $\tan(\pi/2)$ and $\sec(\pi/2)$ are *undefined*. Similarly, the cotangent and cosecant are not defined when $y = 0$. For instance, because $t = 0$ corresponds to $(x, y) = (1, 0)$, cot 0 and csc 0 are *undefined*.

In Figure 4.20, the unit circle has been divided into eight equal arcs, corresponding to t-values of

$$0, \frac{\pi}{4}, \frac{\pi}{2}, \frac{3\pi}{4}, \pi, \frac{5\pi}{4}, \frac{3\pi}{2}, \frac{7\pi}{4}, \text{ and } 2\pi.$$

Similarly, in Figure 4.21, the unit circle has been divided into 12 equal arcs, corresponding to t-values of

$$0, \frac{\pi}{6}, \frac{\pi}{3}, \frac{\pi}{2}, \frac{2\pi}{3}, \frac{5\pi}{6}, \pi, \frac{7\pi}{6}, \frac{4\pi}{3}, \frac{3\pi}{2}, \frac{5\pi}{3}, \frac{11\pi}{6}, \text{ and } 2\pi.$$

Using the (x, y) coordinates in Figures 4.20 and 4.21, you can easily evaluate the exact values of trigonometric functions for common t-values. This procedure is demonstrated in Examples 1 and 2. You should study and learn these exact values for common t-values because they will help you in later sections to perform calculations quickly and easily.

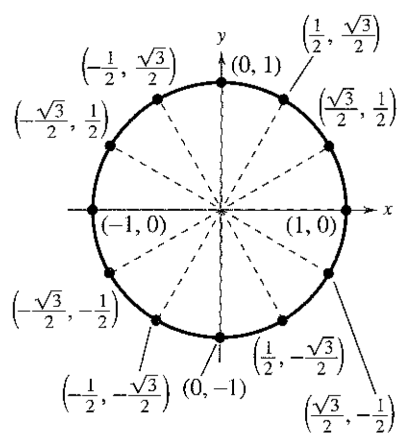


Figure 4.21

Example 1 Evaluating Trigonometric Functions

Evaluate the six trigonometric functions at each real number.

a. $t = \dfrac{\pi}{6}$ **b.** $t = \dfrac{5\pi}{4}$ **c.** $t = 0$ **d.** $t = \pi$

Solution

For each t-value, begin by finding the corresponding point (x, y) on the unit circle. Then use the definitions of trigonometric functions listed on page 260.

a. $t = \pi/6$ corresponds to the point $(x, y) = \left(\sqrt{3}/2, 1/2\right)$.

$$\sin\frac{\pi}{6} = y = \frac{1}{2} \qquad \csc\frac{\pi}{6} = \frac{1}{y} = \frac{1}{1/2} = 2$$

$$\cos\frac{\pi}{6} = x = \frac{\sqrt{3}}{2} \qquad \sec\frac{\pi}{6} = \frac{1}{x} = \frac{2}{\sqrt{3}} = \frac{2\sqrt{3}}{3}$$

$$\tan\frac{\pi}{6} = \frac{y}{x} = \frac{1/2}{\sqrt{3}/2} = \frac{1}{\sqrt{3}} = \frac{\sqrt{3}}{3} \qquad \cot\frac{\pi}{6} = \frac{x}{y} = \frac{\sqrt{3}/2}{1/2} = \sqrt{3}$$

b. $t = 5\pi/4$ corresponds to the point $(x, y) = \left(-\sqrt{2}/2, -\sqrt{2}/2\right)$.

$$\sin\frac{5\pi}{4} = y = -\frac{\sqrt{2}}{2} \qquad \csc\frac{5\pi}{4} = \frac{1}{y} = -\frac{2}{\sqrt{2}} = -\sqrt{2}$$

$$\cos\frac{5\pi}{4} = x = -\frac{\sqrt{2}}{2} \qquad \sec\frac{5\pi}{4} = \frac{1}{x} = -\frac{2}{\sqrt{2}} = -\sqrt{2}$$

$$\tan\frac{5\pi}{4} = \frac{y}{x} = \frac{-\sqrt{2}/2}{-\sqrt{2}/2} = 1 \qquad \cot\frac{5\pi}{4} = \frac{x}{y} = \frac{-\sqrt{2}/2}{-\sqrt{2}/2} = 1$$

c. $t = 0$ corresponds to the point $(x, y) = (1, 0)$.

$$\sin 0 = y = 0 \qquad \csc 0 = \frac{1}{y} \text{ is undefined.}$$

$$\cos 0 = x = 1 \qquad \sec 0 = \frac{1}{x} = \frac{1}{1} = 1$$

$$\tan 0 = \frac{y}{x} = \frac{0}{1} = 0 \qquad \cot 0 = \frac{x}{y} \text{ is undefined.}$$

d. $t = \pi$ corresponds to the point $(x, y) = (-1, 0)$.

$$\sin\pi = y = 0 \qquad \csc\pi = \frac{1}{y} \text{ is undefined.}$$

$$\cos\pi = x = -1 \qquad \sec\pi = \frac{1}{x} = \frac{1}{-1} = -1$$

$$\tan\pi = \frac{y}{x} = \frac{0}{-1} = 0 \qquad \cot\pi = \frac{x}{y} \text{ is undefined.}$$

✓ ***Checkpoint*** Now try Exercise 23.

Example 2 Evaluating Trigonometric Functions

Evaluate the six trigonometric functions at $t = -\dfrac{\pi}{3}$.

Solution

Moving *clockwise* around the unit circle, it follows that $t = -\pi/3$ corresponds to the point $(x, y) = \left(1/2, -\sqrt{3}/2\right)$.

$$\sin\left(-\frac{\pi}{3}\right) = -\frac{\sqrt{3}}{2} \qquad \csc\left(-\frac{\pi}{3}\right) = -\frac{2}{\sqrt{3}} = -\frac{2\sqrt{3}}{3}$$

$$\cos\left(-\frac{\pi}{3}\right) = \frac{1}{2} \qquad \sec\left(-\frac{\pi}{3}\right) = 2$$

$$\tan\left(-\frac{\pi}{3}\right) = \frac{-\sqrt{3}/2}{1/2} = -\sqrt{3} \qquad \cot\left(-\frac{\pi}{3}\right) = \frac{1/2}{-\sqrt{3}/2} = -\frac{1}{\sqrt{3}} = -\frac{\sqrt{3}}{3}$$

✓ *Checkpoint* Now try Exercise 27.

Exploration

With your graphing utility in *radian* and *parametric* modes, enter
X1T = cos T and Y1T = sin T
and use the following settings.
Tmin = 0, Tmax = 6.3,
Tstep = 0.1
Xmin = −1.5, Xmax = 1.5,
Xscl = 1
Ymin = -1, Ymax = 1,
Yscl = 1

1. Graph the entered equations and describe the graph.
2. Use the *trace* feature to move the cursor around the graph. What do the t-values represent? What do the x- and y-values represent?
3. What are the least and greatest values for x and y?

Domain and Period of Sine and Cosine

The *domain* of the sine and cosine functions is the set of all real numbers. To determine the *range* of these two functions, consider the unit circle shown in Figure 4.22. Because $r = 1$, it follows that $\sin t = y$ and $\cos t = x$. Moreover, because (x, y) is on the unit circle, you know that $-1 \le y \le 1$ and $-1 \le x \le 1$. So, the values of sine and cosine also range between -1 and 1.

$$\begin{array}{ccc} -1 \le \;\; y \;\; \le 1 & & -1 \le \;\; x \;\; \le 1 \\ & \text{and} & \\ -1 \le \sin t \le 1 & & -1 \le \cos t \le 1 \end{array}$$

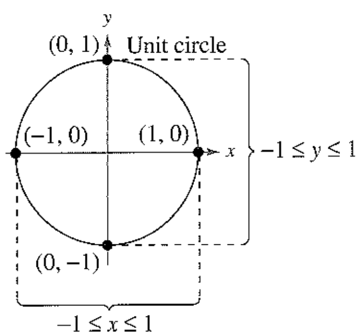


Figure 4.22

Adding 2π to each value of t in the interval $[0, 2\pi]$ completes a second revolution around the unit circle, as shown in Figure 4.23. The values of $\sin(t + 2\pi)$ and $\cos(t + 2\pi)$ correspond to those of $\sin t$ and $\cos t$. Similar results can be obtained for repeated revolutions (positive or negative) around the unit circle. This leads to the general result

$$\sin(t + 2\pi n) = \sin t$$

and

$$\cos(t + 2\pi n) = \cos t$$

for any integer n and real number t. Functions that behave in such a repetitive (or cyclic) manner are called **periodic.**

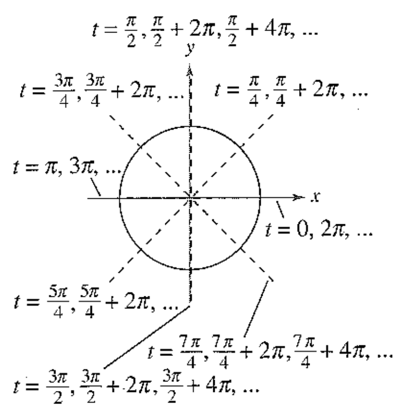


Figure 4.23

> **Definition of a Periodic Function**
>
> A function f is **periodic** if there exists a positive real number c such that
>
> $$f(t + c) = f(t)$$
>
> for all t in the domain of f. The least number c for which f is periodic is called the **period** of f.

Example 3 Using the Period to Evaluate the Sine Function

Evaluate $\sin \frac{13\pi}{6}$ using its period as an aid.

Solution

Because $\frac{13\pi}{6} = 2\pi + \frac{\pi}{6}$, you have

$$\sin \frac{13\pi}{6} = \sin\left(2\pi + \frac{\pi}{6}\right) = \sin \frac{\pi}{6} = \frac{1}{2}.$$

✓ *Checkpoint* Now try Exercise 31.

STUDY TIP

It follows from the definition of periodic function that the sine and cosine functions are periodic and have a period of 2π. The other four trigonometric functions are also periodic, and more will be said about this in Section 4.6.

Recall from Section 1.3 that a function f is *even* if $f(-t) = f(t)$, and it is *odd* if $f(-t) = -f(t)$.

Even and Odd Trigonometric Functions

The cosine and secant functions are *even*.

$\cos(-t) = \cos t$ $\quad\quad$ $\sec(-t) = \sec t$

The sine, cosecant, tangent, and cotangent functions are *odd*.

$\sin(-t) = -\sin t$ $\quad\quad$ $\csc(-t) = -\csc t$

$\tan(-t) = -\tan t$ $\quad\quad$ $\cot(-t) = -\cot t$

Evaluating Trigonometric Functions with a Calculator

When evaluating a trigonometric function with a calculator, you need to set the calculator to the desired *mode* of measurement (degrees or radians).

Most calculators do not have keys for the cosecant, secant, and cotangent functions. To evaluate these functions, you can use the [x^{-1}] key with their respective reciprocal functions sine, cosine, and tangent. For example, to evaluate $\csc(\pi/8)$, use the fact that

$$\csc \frac{\pi}{8} = \frac{1}{\sin(\pi/8)}$$

and enter the following keystroke sequence in radian mode.

[(] [SIN] π [÷] 8 [)] [)] [x^{-1}] [ENTER] $\quad\quad$ Display 2.6131259

TECHNOLOGY TIP

When evaluating trigonometric functions with a calculator, remember to enclose all fractional angle measures in parentheses. For instance, if you want to evaluate $\sin \theta$ for $\theta = \pi/6$, you should enter

[SIN] [(] [π] [÷] 6 [)] [ENTER] .

These keystrokes yield the correct value of 0.5.

Example 4 Using a Calculator

	Function	*Mode*	*Graphing Calculator Keystrokes*	*Display*
a.	$\sin 2\pi/3$	Radian	[SIN] 2 [π] [÷] 3 [)] [ENTER]	0.8660254
b.	$\cot 1.5$	Radian	[(] [TAN] 1.5 [)] [)] [x^{-1}] [ENTER]	0.0709148

✓ *Checkpoint* Now try Exercise 45.

4.2 Exercises

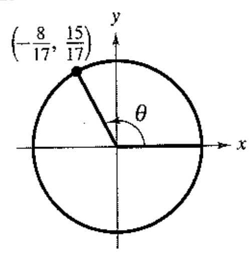
Vocabulary Check

Fill in the blanks.

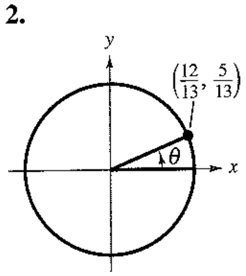
1. Each real number t corresponds to a point (x, y) on the ________ .
2. A function f is ________ if there exists a positive real number c such that $f(t + c) = f(t)$ for all t in the domain of f.
3. A function f is ________ if $f(-t) = -f(t)$ and ________ if $f(-t) = f(t)$.

In Exercises 1–4, determine the exact values of the six trigonometric functions of the angle θ.

1.



2.



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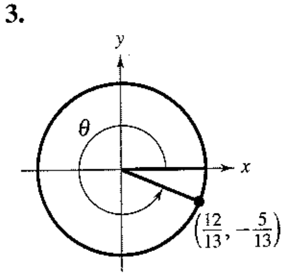

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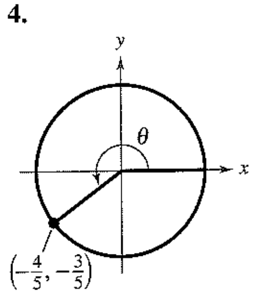


In Exercises 5–12, find the point (x, y) on the unit circle that corresponds to the real number t.

5. $t = \frac{\pi}{4}$
6. $t = \frac{\pi}{3}$
7. $t = \frac{7\pi}{6}$
8. $t = \frac{5\pi}{4}$
9. $t = \frac{2\pi}{3}$
10. $t = \frac{5\pi}{3}$
11. $t = \frac{3\pi}{2}$
12. $t = \pi$

In Exercises 13–22, evaluate (if possible) the sine, cosine, and tangent of the real number.

13. $t = \frac{\pi}{4}$
14. $t = \frac{\pi}{3}$
15. $t = -\frac{\pi}{6}$
16. $t = -\frac{3\pi}{4}$
17. $t = -\frac{7\pi}{4}$
18. $t = -\frac{4\pi}{3}$
19. $t = \frac{5\pi}{3}$
20. $t = \frac{11\pi}{6}$
21. $t = -\frac{3\pi}{2}$
22. $t = -2\pi$

In Exercises 23–28, evaluate (if possible) the six trigonometric functions of the real number.

23. $t = \frac{3\pi}{4}$
24. $t = \frac{5\pi}{6}$
25. $t = \frac{\pi}{2}$
26. $t = \frac{3\pi}{2}$
27. $t = -\frac{2\pi}{3}$
28. $t = -\frac{7\pi}{4}$

In Exercises 29–36, evaluate the trigonometric function using its period as an aid.

29. $\sin 5\pi$
30. $\cos 7\pi$
31. $\cos \frac{8\pi}{3}$
32. $\sin \frac{9\pi}{4}$
33. $\cos\left(-\frac{13\pi}{6}\right)$
34. $\sin\left(-\frac{19\pi}{6}\right)$
35. $\sin\left(-\frac{9\pi}{4}\right)$
36. $\cos\left(-\frac{8\pi}{3}\right)$

In Exercises 37–42, use the value of the trigonometric function to evaluate the indicated functions.

37. $\sin t = \frac{1}{3}$
    (a) $\sin(-t)$
    (b) $\csc(-t)$
38. $\cos t = -\frac{3}{4}$
    (a) $\cos(-t)$
    (b) $\sec(-t)$

39. $\cos(-t) = -\frac{1}{5}$
(a) $\cos t$
(b) $\sec(-t)$

40. $\sin(-t) = \frac{3}{8}$
(a) $\sin t$
(b) $\csc t$

41. $\sin t = \frac{4}{5}$
(a) $\sin(\pi - t)$
(b) $\sin(t + \pi)$

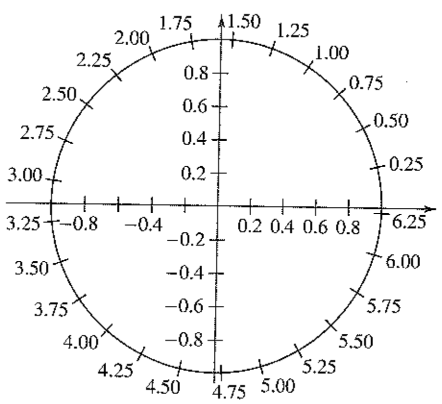
42. $\cos t = \frac{4}{5}$
(a) $\cos(\pi - t)$
(b) $\cos(t + \pi)$

In Exercises 43–52, use a calculator to evaluate the expression. Round your answer to four decimal places.

43. $\sin \frac{\pi}{6}$ **44.** $\tan \frac{\pi}{2}$
45. $\csc 1.3$ **46.** $\cot 3.7$
47. $\cos(-1.7)$ **48.** $\cos(-2.5)$
49. $\csc 0.8$ **50.** $\sec 1.8$
51. $\sec 22.8$ **52.** $\sin(-13.4)$

Estimation **In Exercises 53 and 54, use the figure and a straightedge to approximate the value of each trigonometric function. Check your approximation using a graphing utility. To print an enlarged copy of the graph, go to the website *www.mathgraphs.com*.**

53. (a) $\sin 5$ (b) $\cos 2$
54. (a) $\sin 0.75$ (b) $\cos 2.5$



Estimation **In Exercises 55 and 56, use the figure in Exercises 53 and 54 and a straightedge to approximate the solution of each equation, where $0 \le t < 2\pi$. Check your approximation using a graphing utility. To print an enlarged copy of the graph, go to the website *www.mathgraphs.com*.**

55. (a) $\sin t = 0.25$ (b) $\cos t = -0.25$
56. (a) $\sin t = -0.75$ (b) $\cos t = 0.75$

57. ***Electric Circuits*** The initial current and charge in an electrical circuit are zero. The current when 100 volts is applied to the circuit is given by

$$I = 5e^{-2t} \sin t$$

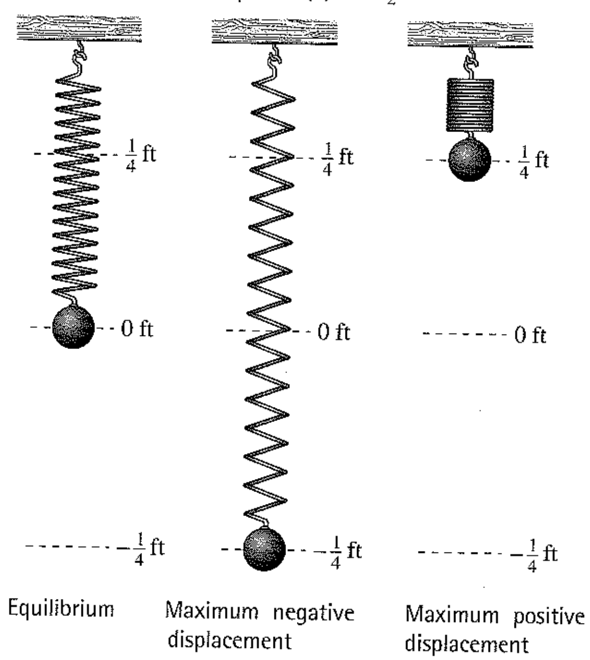
where the resistance, inductance, and capacitance are 80 ohms, 20 henrys, and 0.01 farad, respectively. Approximate the current (in amperes) $t = 0.7$ second after the voltage is applied.

58. ***Electrical Circuits*** Approximate the current (in amperes) in the electrical circuit in Exercise 57 $t = 1.4$ seconds after the voltage is applied.

59. ***Harmonic Motion*** The displacement from equilibrium of an oscillating weight suspended by a spring is given by

$$y(t) = \frac{1}{4} \cos 6t$$

where y is the displacement in feet and t is the time in seconds (see figure). Find the displacement when (a) $t = 0$, (b) $t = \frac{1}{4}$, and (c) $t = \frac{1}{2}$.



Simple Harmonic Motion

60. ***Harmonic Motion*** The displacement from equilibrium of an oscillating weight suspended by a spring and subject to the damping effect of friction is given by $y(t) = \frac{1}{4}e^{-t} \cos 6t$, where y is the displacement (in feet) and t is the time (in seconds). Find the displacement when (a) $t = 0$, (b) $t = \frac{1}{4}$, and (c) $t = \frac{1}{2}$.

Synthesis

True or False? **In Exercises 61–64, determine whether the statement is true or false. Justify your answer.**

61. Because $\sin(-t) = -\sin t$, it can be said that the sine of a negative angle is a negative number.

62. $\sin a = \sin(a - 6\pi)$

63. The real number 0 corresponds to the point $(0, 1)$ on the real number line.

64. $\cos\left(-\frac{7\pi}{2}\right) = \cos\left(\pi + \frac{\pi}{2}\right)$

65. ***Exploration*** Let (x_1, y_1) and (x_2, y_2) be points on the unit circle corresponding to $t = t_1$ and $t = \pi - t_1$, respectively.

(a) Identify the symmetry of the points (x_1, y_1) and (x_2, y_2).

(b) Make a conjecture about any relationship between $\sin t_1$ and $\sin(\pi - t_1)$.

(c) Make a conjecture about any relationship between $\cos t_1$ and $\cos(\pi - t_1)$.

66. ***Exploration*** Let (x_1, y_1) and (x_2, y_2) be points on the unit circle corresponding to $t = t_1$ and $t = t_1 + \pi$, respectively.

(a) Identify the symmetry of the points (x_1, y_1) and (x_2, y_2).

(b) Make a conjecture about any relationship between $\sin t_1$ and $\sin(t_1 + \pi)$.

(c) Make a conjecture about any relationship between $\cos t_1$ and $\cos(t_1 + \pi)$.

67. Verify that $\cos 2t \neq 2 \cos t$ by approximating $\cos 1.5$ and $2 \cos 0.75$.

68. Verify that $\sin(t_1 + t_2) \neq \sin t_1 + \sin t_2$ by approximating $\sin 0.25$, $\sin 0.75$, and $\sin 1$.

69. Use the unit circle to verify that the cosine and secant functions are even.

70. Use the unit circle to verify that the sine, cosecant, tangent, and cotangent functions are odd.

71. ***Think About It*** Because $f(t) = \sin t$ is an odd function and $g(t) = \cos t$ is an even function, what can be said about the function $h(t) = f(t)g(t)$?

72. ***Think About It*** Because $f(t) = \sin t$ and $g(t) = \tan t$ are odd functions, what can be said about the function $h(t) = f(t)g(t)$?

Review

In Exercises 73–76, find the inverse function f^{-1} of the one-to-one function f. Use a graphing utility to graph both f and f^{-1} in the same viewing window.

73. $f(x) = \frac{1}{2}(3x - 2)$

74. $f(x) = \frac{1}{4}x^3 + 1$

75. $f(x) = \sqrt{x^2 - 4}, \quad x \geq 2$

76. $f(x) = \frac{2x}{x + 1}, \quad x > -1$

In Exercises 77–80, sketch the graph of the rational function by hand. Show all asymptotes. Use a graphing utility to verify your graph.

77. $f(x) = \frac{2x}{x - 3}$

78. $f(x) = \frac{5x}{x^2 + x - 6}$

79. $f(x) = \frac{x^2 + 3x - 10}{2x^2 - 8}$

80. $f(x) = \frac{x^3 - 6x^2 + x - 1}{2x^2 - 5x - 8}$

81. ***Average Cost*** The average cost $\overline{C}$ (in dollars per pound) of recycling a waste product x (in pounds) is given by

$$\overline{C} = \frac{450{,}000 + 5x}{x}, \quad x > 0.$$

Find the average cost of recycling $x = 10{,}000$ pounds, $x = 100{,}000$ pounds, and $x = 1{,}000{,}000$ pounds. According to this model, what is the limiting average cost as the number of pounds increases?

82. ***Inflation*** If the inflation rate averages 4.5% over the next 10 years, the approximate cost C of goods or services t years from now is given by

$$C(t) = P(1.045)^t$$

where P is the present cost. The price of a tire is presently \$69.95. Estimate the price 10 years from now.

83. ***Human Memory Model*** Participants in an industrial psychology study were taught a simple mechanical task and then tested monthly on this task for a period of 1 year. The average score for the class is given by

$$f(t) = 95 - 12 \log_{10}(t + 1), \quad 0 \leq t \leq 12$$

where t is the time in months. What was the average score after 3 months and after 6 months?

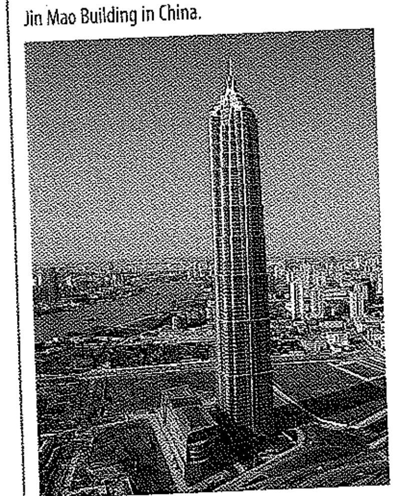
4.3 Right Triangle Trigonometry

What you should learn

- Evaluate trigonometric functions of acute angles.
- Use the fundamental trigonometric identities.
- Use a calculator to evaluate trigonometric functions.
- Use trigonometric functions to model and solve real-life problems.

Why you should learn it

You can use trigonometry to analyze all aspects of a geometric figure. For instance, Exercise 62 on page 276 shows you how trigonometric functions can be used to approximate the height of the Jin Mao Building in China.



Chen Yixan/China Stock

The Six Trigonometric Functions

Our second look at the trigonometric functions is from a *right triangle* perspective. Consider a right triangle, with one acute angle labeled θ, as shown in Figure 4.24. Relative to the angle θ, the three sides of the triangle are the **hypotenuse,** the **opposite side** (the side opposite the angle θ), and the **adjacent side** (the side adjacent to the angle θ).

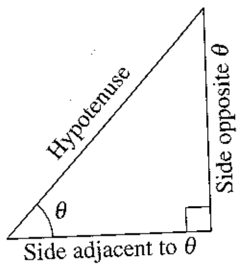


Figure 4.24

Using the lengths of these three sides, you can form six ratios that define the six trigonometric functions of the acute angle θ.

sine	**cosecant**
cosine	**secant**
tangent	**cotangent**

In the following definitions it is important to see that $0° < \theta < 90°$ (θ lies in the first quadrant) and that for such angles the value of each trigonometric function is *positive*.

Right Triangle Definitions of Trigonometric Functions

Let θ be an *acute* angle of a right triangle. Then the six trigonometric functions *of the angle* θ are defined as follows. (Note that the functions in the second row are the *reciprocals* of the corresponding functions in the first row.)

$$\sin\theta = \frac{\text{opp}}{\text{hyp}} \qquad \cos\theta = \frac{\text{adj}}{\text{hyp}} \qquad \tan\theta = \frac{\text{opp}}{\text{adj}}$$

$$\csc\theta = \frac{\text{hyp}}{\text{opp}} \qquad \sec\theta = \frac{\text{hyp}}{\text{adj}} \qquad \cot\theta = \frac{\text{adj}}{\text{opp}}$$

The abbreviations "opp", "adj", and "hyp" represent the lengths of the three sides of a right triangle.

opp = the length of the side *opposite* θ

adj = the length of the side *adjacent* to θ

hyp = the length of the *hypotenuse*

Example 1 Evaluating Trigonometric Functions

Use the triangle in Figure 4.25 to find the exact values of the six trigonometric functions of θ.

Solution

By the Pythagorean Theorem, $(\text{hyp})^2 = (\text{opp})^2 + (\text{adj})^2$, it follows that

$$\text{hyp} = \sqrt{4^2 + 3^2} = \sqrt{25} = 5.$$

So, the six trigonometric functions of θ are

$$\sin\theta = \frac{\text{opp}}{\text{hyp}} = \frac{4}{5} \qquad \csc\theta = \frac{\text{hyp}}{\text{opp}} = \frac{5}{4}$$

$$\cos\theta = \frac{\text{adj}}{\text{hyp}} = \frac{3}{5} \qquad \sec\theta = \frac{\text{hyp}}{\text{adj}} = \frac{5}{3}$$

$$\tan\theta = \frac{\text{opp}}{\text{adj}} = \frac{4}{3} \qquad \cot\theta = \frac{\text{adj}}{\text{opp}} = \frac{3}{4}$$

✓ *Checkpoint* Now try Exercise 3.

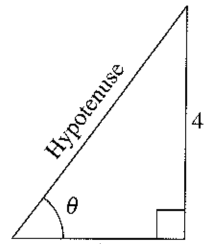


Figure 4.25

In Example 1, you were given the lengths of two sides of the right triangle, but not the angle θ. Often you will be asked to find the trigonometric functions for a *given* acute angle θ. To do this, you can construct a right triangle having θ as one of its angles.

Example 2 Evaluating Trigonometric Functions of 45°

Find the exact values of sin 45°, cos 45°, and tan 45°.

Solution

Construct a right triangle having 45° as one of its acute angles, as shown in Figure 4.26. Choose 1 as the length of the adjacent side. From geometry, you know that the other acute angle is also 45°. So, the triangle is isosceles and the length of the opposite side is also 1. Using the Pythagorean Theorem, you find the length of the hypotenuse to be $\sqrt{2}$.

$$\sin 45^\circ = \frac{\text{opp}}{\text{hyp}} = \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2}$$

$$\cos 45^\circ = \frac{\text{adj}}{\text{hyp}} = \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2}$$

$$\tan 45^\circ = \frac{\text{opp}}{\text{adj}} = \frac{1}{1} = 1$$

✓ *Checkpoint* Now try Exercise 17.

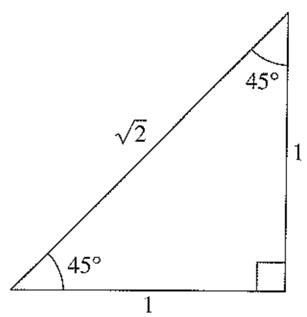


Figure 4.26

TECHNOLOGY TIP You can use a calculator to convert the answers in Example 2 to decimals. However, the radical form is the exact value and in most cases, the exact value is preferred.

Example 3 Evaluating Trigonometric Functions of 30° and 60°

Use the equilateral triangle shown in Figure 4.27 to find the exact values of sin 60°, cos 60°, sin 30°, and cos 30°.

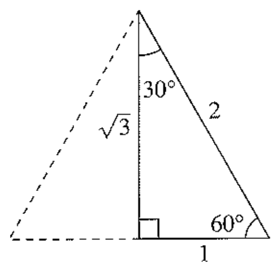


Figure 4.27

Solution

Use the Pythagorean Theorem and the equilateral triangle to verify the lengths of the sides given in Figure 4.27. For $\theta = 60°$, you have adj $= 1$, opp $= \sqrt{3}$, and hyp $= 2$. So,

$$\sin 60° = \frac{\text{opp}}{\text{hyp}} = \frac{\sqrt{3}}{2} \quad \text{and} \quad \cos 60° = \frac{\text{adj}}{\text{hyp}} = \frac{1}{2}.$$

For $\theta = 30°$, adj $= \sqrt{3}$, opp $= 1$, and hyp $= 2$. So,

$$\sin 30° = \frac{\text{opp}}{\text{hyp}} = \frac{1}{2} \quad \text{and} \quad \cos 30° = \frac{\text{adj}}{\text{hyp}} = \frac{\sqrt{3}}{2}.$$

✓ ***Checkpoint*** Now try Exercise 19.

STUDY TIP

Because the angles 30°, 45°, and 60° ($\pi/6$, $\pi/4$, and $\pi/3$) occur frequently in trigonometry, you should learn to construct the triangles shown in Figures 4.26 and 4.27.

Sines, Cosines, and Tangents of Special Angles

$\sin 30° = \sin \frac{\pi}{6} = \frac{1}{2}$	$\cos 30° = \cos \frac{\pi}{6} = \frac{\sqrt{3}}{2}$	$\tan 30° = \tan \frac{\pi}{6} = \frac{\sqrt{3}}{3}$
$\sin 45° = \sin \frac{\pi}{4} = \frac{\sqrt{2}}{2}$	$\cos 45° = \cos \frac{\pi}{4} = \frac{\sqrt{2}}{2}$	$\tan 45° = \tan \frac{\pi}{4} = 1$
$\sin 60° = \sin \frac{\pi}{3} = \frac{\sqrt{3}}{2}$	$\cos 60° = \cos \frac{\pi}{3} = \frac{1}{2}$	$\tan 60° = \tan \frac{\pi}{3} = \sqrt{3}$

In the box, note that $\sin 30° = \frac{1}{2} = \cos 60°$. This occurs because 30° and 60° are complementary angles, and, in general, it can be shown from the right triangle definitions that *cofunctions of complementary angles are equal.* That is, if θ is an acute angle, the following relationships are true.

$$\sin(90° - \theta) = \cos\theta \qquad \cos(90° - \theta) = \sin\theta$$

$$\tan(90° - \theta) = \cot\theta \qquad \cot(90° - \theta) = \tan\theta$$

$$\sec(90° - \theta) = \csc\theta \qquad \csc(90° - \theta) = \sec\theta$$

Trigonometric Identities

In trigonometry, a great deal of time is spent studying relationships between trigonometric functions (identities).

> **Exploration**
>
> Select a number t and use your graphing utility to calculate $(\sin t)^2 + (\cos t)^2$. Repeat this experiment for other values of t and explain why the answer is always the same. Is the result true in both *radian* and *degree* modes?

Fundamental Trigonometric Identities

Reciprocal Identities

$$\sin\theta = \frac{1}{\csc\theta} \qquad \cos\theta = \frac{1}{\sec\theta} \qquad \tan\theta = \frac{1}{\cot\theta}$$

$$\csc\theta = \frac{1}{\sin\theta} \qquad \sec\theta = \frac{1}{\cos\theta} \qquad \cot\theta = \frac{1}{\tan\theta}$$

Quotient Identities

$$\tan\theta = \frac{\sin\theta}{\cos\theta} \qquad \cot\theta = \frac{\cos\theta}{\sin\theta}$$

Pythagorean Identities

$$\sin^2\theta + \cos^2\theta = 1$$

$$1 + \tan^2\theta = \sec^2\theta$$

$$1 + \cot^2\theta = \csc^2\theta$$

Note that $\sin^2\theta$ represents $(\sin\theta)^2$, $\cos^2\theta$ represents $(\cos\theta)^2$, and so on.

Example 4 Applying Trigonometric Identities

Let θ be an acute angle such that $\sin\theta = 0.6$. Find the values of (a) $\cos\theta$ and (b) $\tan\theta$ using trigonometric identities.

Solution

a. To find the value of $\cos\theta$, use the Pythagorean identity

$$\sin^2\theta + \cos^2\theta = 1.$$

So, you have

$$(0.6)^2 + \cos^2\theta = 1 \qquad \text{Substitute 0.6 for } \sin\theta.$$

$$\cos^2\theta = 1 - (0.6)^2 = 0.64 \qquad \text{Subtract } (0.6)^2 \text{ from each side.}$$

$$\cos\theta = \sqrt{0.64} = 0.8. \qquad \text{Extract positive square root.}$$

b. Now, knowing the sine and cosine of θ, you can find the tangent of θ to be

$$\tan\theta = \frac{\sin\theta}{\cos\theta} = \frac{0.6}{0.8} = 0.75.$$

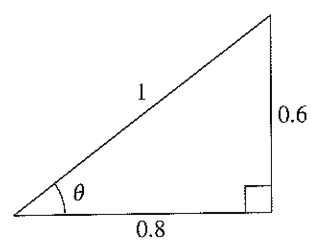


Figure 4.28

Use the definitions of $\cos\theta$ and $\tan\theta$, and the triangle shown in Figure 4.28, to check these results.

✓ *Checkpoint* Now try Exercise 29.

Example 5 Using Trigonometric Identities

Use trigonometric identities to transform one side of the equation into the other $(0 < \theta < \pi/2)$.

a. $\cos\theta \sec\theta = 1$ **b.** $(\sec\theta + \tan\theta)(\sec\theta - \tan\theta) = 1$

Solution

Simplify the expression on the left-hand side of the equation until you obtain the right-hand side.

a. $\cos\theta \sec\theta = \left(\frac{1}{\sec\theta}\right)\sec\theta$ Reciprocal identity

$= 1$ Divide out common factor.

b. $(\sec\theta + \tan\theta)(\sec\theta - \tan\theta)$

$= \sec^2\theta - \sec\theta\tan\theta + \sec\theta\tan\theta - \tan^2\theta$ Distributive Property

$= \sec^2\theta - \tan^2\theta$ Simplify.

$= 1$ Pythagorean identity

✓ *Checkpoint* Now try Exercise 39.

Evaluating Trigonometric Functions with a Calculator

To use a calculator to evaluate trigonometric functions of angles measured in degrees, set the calculator to *degree* mode and then proceed as demonstrated in Section 4.2.

Example 6 Using a Calculator

Use a calculator to evaluate $\sec(5° 40' 12'')$.

Solution

Begin by converting to decimal degree form.

$$5° 40' 12'' = 5° + \left(\frac{40}{60}\right)^{\circ} + \left(\frac{12}{3600}\right)^{\circ} = 5.67°$$

Then use a calculator in *degree* mode to evaluate sec 5.67°.

Function	*Graphing Calculator Keystrokes*	*Display*
$\sec(5°40'12'') = \sec 5.67°$	(COS 5.67) x^{-1} ENTER	1.0049166

✓ *Checkpoint* Now try Exercise 43.

STUDY TIP

Remember that throughout this text, it is assumed that angles are measured in radians unless noted otherwise. For example, sin 1 means the sine of 1 radian and sin 1° means the sine of 1 degree.

Applications Involving Right Triangles

Many applications of trigonometry involve a process called **solving right triangles.** In this type of application, you are usually given one side of a right triangle and one of the acute angles and asked to find one of the other sides, *or* you are given two sides and asked to find one of the acute angles.

In Example 7, the angle you are given is the **angle of elevation,** which represents the angle from the horizontal upward to the object. In other applications you may be given the **angle of depression,** which represents the angle from the horizontal downward to the object.

Example 7 Using Trigonometry to Solve a Right Triangle

A surveyor is standing 50 feet from the base of a large tree, as shown in Figure 4.29. The surveyor measures the angle of elevation to the top of the tree as 71.5°. How tall is the tree?

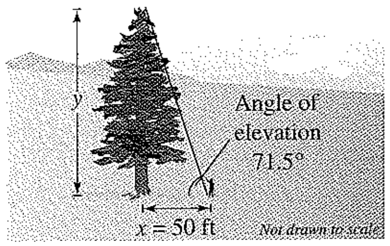


Figure 4.29

Algebraic Solution

From Figure 4.29, you can see that

$$\tan 71.5° = \frac{\text{opp}}{\text{adj}} = \frac{y}{x}$$

where $x = 50$ and y is the height of the tree. So, the height of the tree is

$$y = x \tan 71.5°$$
$$\approx 50(2.9887)$$
$$\approx 149.4 \text{ feet.}$$

☑ ***Checkpoint*** Now try Exercise 59.

Example 8 Using Trigonometry to Solve a Right Triangle

You are 200 yards from a river. Rather than walking directly to the river, you walk 400 yards along a straight path to the river's edge. Find the acute angle θ between this path and the river's edge, as illustrated in Figure 4.30.

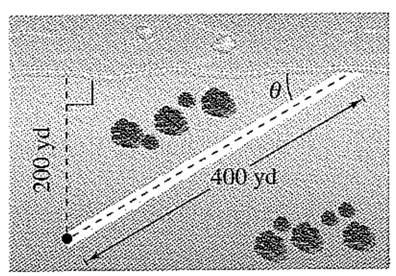


Figure 4.30

Solution

From Figure 4.30, you can see that the sine of the angle θ is

$$\sin \theta = \frac{\text{opp}}{\text{hyp}} = \frac{200}{400} = \frac{1}{2}.$$

Now you should recognize that $\theta = 30°$.

☑ ***Checkpoint*** Now try Exercise 61.

In Example 8, you were able to recognize that $\theta = 30°$ is the acute angle that satisfies the equation $\sin \theta = \frac{1}{2}$. Suppose, however, that you were given the equation $\sin \theta = 0.6$ and were asked to find the acute angle θ. Because

$$\sin 30° = \frac{1}{2} = 0.5000$$

and

$$\sin 45° = \frac{1}{\sqrt{2}} \approx 0.7071,$$

you might guess that θ lies somewhere between 30° and 45°. In a later section, you will study a method by which a more precise value of θ can be determined.

TECHNOLOGY TIP

Calculators and graphing utilities have both *degree* and *radian* modes. As you progress through this chapter, be sure you use the correct mode.

Example 9 Solving a Right Triangle

Find the length c of the skateboard ramp shown in Figure 4.31.

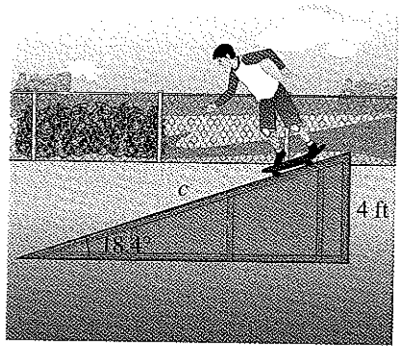


Figure 4.31

Solution

From Figure 4.31, you can see that

$$\sin 18.4° = \frac{\text{opp}}{\text{hyp}} = \frac{4}{c}.$$

So, the length of the ramp is

$$c = \frac{4}{\sin 18.4°} \approx \frac{4}{0.3156} \approx 12.7 \text{ feet.}$$

✓ ***Checkpoint*** Now try Exercise 63.

4.3 Exercises

Vocabulary Check

1. Match the trigonometric function with its right triangle definition.

(a) sine (b) cosine (c) tangent

(d) cosecant (e) secant (f) cotangent

(i) $\frac{\text{hyp}}{\text{adj}}$ (ii) $\frac{\text{opp}}{\text{adj}}$ (iii) $\frac{\text{opp}}{\text{hyp}}$

(iv) $\frac{\text{adj}}{\text{opp}}$ (v) $\frac{\text{hyp}}{\text{opp}}$ (vi) $\frac{\text{adj}}{\text{hyp}}$

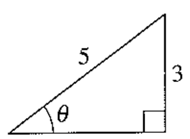
In Exercises 2 and 3, fill in the blanks.

2. Relative to the acute angle θ, the three sides of a right triangle are the _______, the _______ side, and the _______ side.

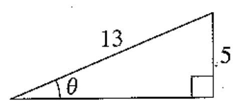
3. An angle that measures from the horizontal upward to an object is called the angle of _______, whereas an angle that measures from the horizontal downward to an object is called the angle of _______ .

In Exercises 1–4, find the exact values of the six trigonometric functions of the angle θ shown in the figure. (Use the Pythagorean Theorem to find the third side of the triangle.)

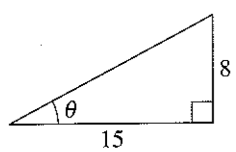
1.



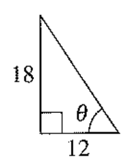
2.



3.

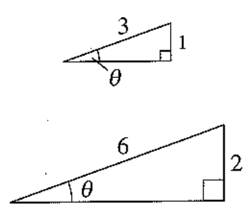


4.

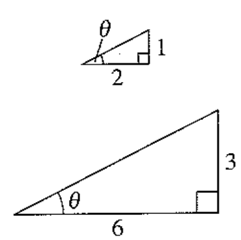


In Exercises 5–8, find the exact values of the six trigonometric functions of the angle θ for each of the triangles. Explain why the function values are the same.

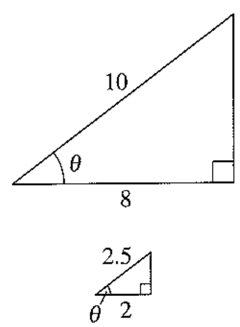
5.



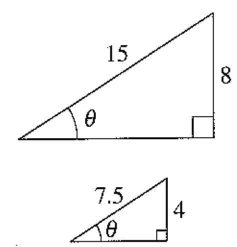
6.



7.



8.



In Exercises 9–16, sketch a right triangle corresponding to the trigonometric function of the acute angle θ. Use the Pythagorean Theorem to determine the third side of the triangle and then find the other five trigonometric functions of θ.

9. $\sin\theta = \frac{5}{6}$ **10.** $\cot\theta = 5$

11. $\sec\theta = 4$ **12.** $\cos\theta = \frac{3}{7}$

13. $\tan\theta = 3$ **14.** $\csc\theta = \frac{17}{4}$

15. $\cot\theta = \frac{9}{4}$ **16.** $\sin\theta = \frac{3}{8}$

In Exercises 17–26, construct an appropriate triangle to complete the table. ($0 \le \theta \le 90°, 0 \le \theta \le \pi/2$)

	Function	θ (deg)	θ (rad)	*Function Value*
17.	sin	30°		
18.	cos	45°		

	Function	*θ (deg)*	*θ (rad)*	*Function Value*
19.	tan		$\frac{\pi}{3}$	
20.	sec		$\frac{\pi}{4}$	
21.	cot			$\frac{\sqrt{3}}{3}$
22.	csc			$\sqrt{2}$
23.	cos		$\frac{\pi}{6}$	
24.	sin		$\frac{\pi}{4}$	
25.	cot			1
26.	tan			$\frac{1}{\sqrt{3}}$

In Exercises 27–32, use the given function value(s) and the trigonometric identities to find the indicated trigonometric functions.

27. $\sin 60° = \frac{\sqrt{3}}{2}$, $\cos 60° = \frac{1}{2}$

(a) $\tan 60°$ (b) $\sin 30°$

(c) $\cos 30°$ (d) $\cot 60°$

28. $\sin 30° = \frac{1}{2}$, $\tan 30° = \frac{\sqrt{3}}{3}$

(a) $\csc 30°$ (b) $\cot 60°$

(c) $\cos 30°$ (d) $\cot 30°$

29. $\csc \theta = 3$, $\sec \theta = \frac{3\sqrt{2}}{4}$

(a) $\sin \theta$ (b) $\cos \theta$

(c) $\tan \theta$ (d) $\sec(90° - \theta)$

30. $\sec \theta = 5$, $\tan \theta = 2\sqrt{6}$

(a) $\cos \theta$ (b) $\cot \theta$

(c) $\cot(90° - \theta)$ (d) $\sin \theta$

31. $\cos \alpha = \frac{1}{4}$

(a) $\sec \alpha$ (b) $\sin \alpha$

(c) $\cot \alpha$ (d) $\sin(90° - \alpha)$

32. $\tan \beta = 5$

(a) $\cot \beta$ (b) $\cos \beta$

(c) $\tan(90° - \beta)$ (d) $\csc \beta$

In Exercises 33–40, use trigonometric identities to transform one side of the equation into the other ($0 < \theta < \pi/2$).

33. $\tan \theta \cot \theta = 1$ **34.** $\csc \theta \tan \theta = \sec \theta$

35. $\tan \theta \cos \theta = \sin \theta$ **36.** $\cot \theta \sin \theta = \cos \theta$

37. $(1 + \cos \theta)(1 - \cos \theta) = \sin^2 \theta$

38. $(\csc \theta + \cot \theta)(\csc \theta - \cot \theta) = 1$

39. $\frac{\sin \theta}{\cos \theta} + \frac{\cos \theta}{\sin \theta} = \csc \theta \sec \theta$

40. $\frac{\tan \theta + \cot \theta}{\tan \theta} = \csc^2 \theta$

In Exercises 41–46, use a calculator to evaluate each function. Round your answers to four decimal places. (Be sure the calculator is in the correct angle mode.)

41. (a) $\sin 41°$ (b) $\cos 87°$

42. (a) $\tan 18.5°$ (b) $\cot 71.5°$

43. (a) $\sec 42° 12'$ (b) $\csc 48° 7'$

44. (a) $\cos 8° 50' 25''$ (b) $\sec 8° 50' 25''$

45. (a) $\cot \frac{\pi}{16}$ (b) $\tan \frac{\pi}{8}$

46. (a) $\sec 1.54$ (b) $\cos 1.25$

In Exercises 47–52, find each value of θ in degrees ($0° < \theta < 90°$) and radians ($0 < \theta < \pi/2$) without using a calculator.

47. (a) $\sin \theta = \frac{1}{2}$ (b) $\csc \theta = 2$

48. (a) $\cos \theta = \frac{\sqrt{2}}{2}$ (b) $\tan \theta = 1$

49. (a) $\sec \theta = 2$ (b) $\cot \theta = 1$

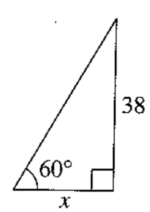
50. (a) $\tan \theta = \sqrt{3}$ (b) $\cos \theta = \frac{1}{2}$

51. (a) $\csc \theta = \frac{2\sqrt{3}}{3}$ (b) $\sin \theta = \frac{\sqrt{2}}{2}$

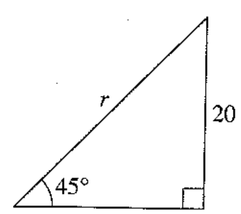
52. (a) $\cot \theta = \frac{\sqrt{3}}{3}$ (b) $\sec \theta = \sqrt{2}$

In Exercises 53–56, solve for *x*, *y*, or *r*, as indicated.

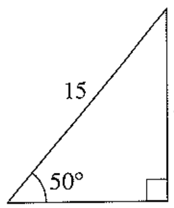
53. Solve for *x*.



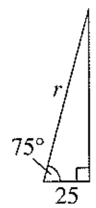
54. Solve for *r*.



55. Solve for y.



56. Solve for r.



57. ***Height*** A six-foot person walks from the base of a streetlight directly toward the tip of the shadow cast by the streetlight. When the person is 16 feet from the streetlight and 5 feet from the tip of the streetlight's shadow, the person's shadow starts to appear beyond the streetlight's shadow.

(a) Draw a right triangle that gives a visual representation of the problem. Show the known quantities of the triangle and use a variable to indicate the height of the streetlight.

(b) Use a trigonometric function to write an equation involving the unknown quantity.

(c) What is the height of the streetlight?

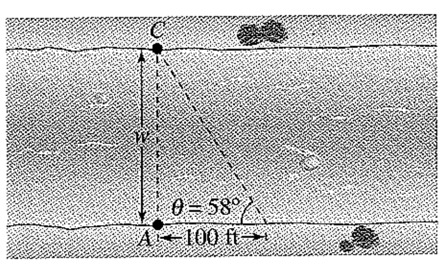
58. ***Height*** A 30-meter line is used to tether a helium-filled balloon. Because of a breeze, the line makes an angle of approximately 75° with the ground.

(a) Draw a right triangle that gives a visual representation of the problem. Show the known quantities of the triangle and use a variable to indicate the height of the balloon.

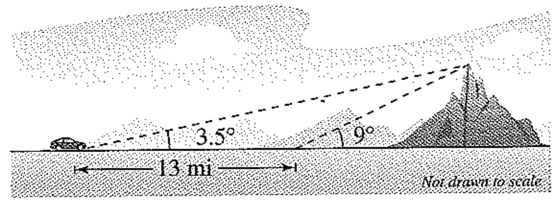
(b) Use a trigonometric function to write an equation involving the unknown quantity.

(c) What is the height of the balloon?

59. ***Width*** A biologist wants to know the width w of a river in order to properly set instruments for studying the pollutants in the water. From point A, the biologist walks downstream 100 feet and sights to point C. From this sighting, it is determined that $\theta = 58°$. How wide is the river? Verify your result numerically.



60. ***Height of a Mountain*** In traveling across flat land you notice a mountain directly in front of you. Its angle of elevation (to the peak) is 3.5°. After you drive 13 miles closer to the mountain, the angle of elevation is 9° (see figure). Approximate the height of the mountain.



61. ***Angle of Elevation*** A ramp 20 feet in length rises to a loading platform that is $3\frac{1}{3}$ feet off the ground.

(a) Draw a right triangle that gives a visual representation of the problem. Show the known quantities of the triangle and use a variable to indicate the angle of elevation of the ramp.

(b) Use a trigonometric function to write an equation involving the unknown quantity.

(c) Use a graphing utility to approximate the angle of elevation numerically.

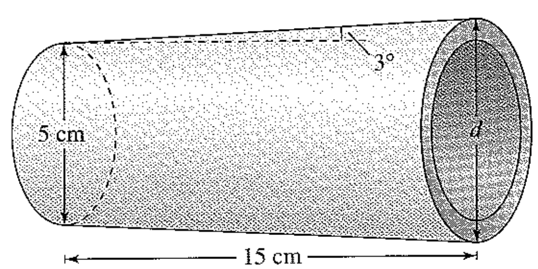
62. ***Jin Mao Building*** You are standing 65 meters from the base of the Jin Mao Building in Shanghai, China. You estimate that the angle of elevation to the top of the 88th floor (sightseeing level) is 80°. What is the approximate height of the building? One of your friends is on the sightseeing level. What is the distance between you and your friend?

63. ***Length*** A guywire is stretched from the top of a 200-foot broadcasting tower to an anchor making an angle of 58° with the ground.

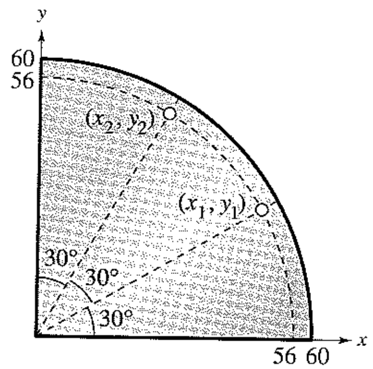
(a) How long is the wire?

(b) How far is the anchor from the base of the tower?

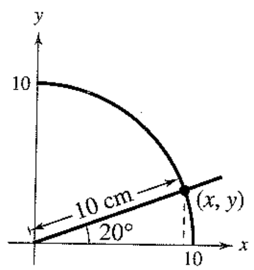
64. ***Machine Shop Calculations*** A tapered shaft has a diameter of 5 centimeters at the small end and is 15 centimeters long (see figure).The taper is 3°. Find the diameter d of the large end of the shaft.



65. ***Machine Shop Calculations*** A steel plate has the form of one fourth of a circle with a radius of 60 centimeters. Two two-centimeter holes are to be drilled in the plate, positioned as shown in the figure. Find the coordinates of the center of each hole.



66. ***Geometry*** Use a compass to sketch a quarter of a circle of radius 10 centimeters. Using a protractor, construct an angle of 20° in standard position (see figure). Drop a perpendicular from the point of intersection of the terminal side of the angle and the arc of the circle. By actual measurement, calculate the coordinates (x, y) of the point of intersection and use these measurements to approximate the six trigonometric functions of a 20° angle.



Synthesis

True or False? **In Exercises 67–69, determine whether the statement is true or false. Justify your answer.**

67. $\sin 60° \csc 60° = 1$

68. $\sin 45° + \cos 45° = 1$

69. $\cot^2 10° - \csc^2 10° = -1$

70. ***Think About It*** You are given the value $\tan \theta$. Is it possible to find the value of $\sec \theta$ without finding the measure of θ? Explain.

71. ***Exploration***

(a) Use a graphing utility to complete the table. Round your results to four decimal places.

θ	0°	20°	40°	60°	80°
$\sin \theta$					
$\cos \theta$					
$\tan \theta$					

(b) Classify each of the three trigonometric functions as increasing or decreasing for the table values.

(c) From the values in the table, verify that the tangent function is the quotient of the sine and cosine functions.

72. ***Exploration*** Use a graphing utility to complete the table and make a conjecture about the relationship between $\cos \theta$ and $\sin(90° - \theta)$. What are the angles θ and $90° - \theta$ called?

θ	0°	20°	40°	60°	80°
$\cos \theta$					
$\sin (90° - \theta)$					

Review

In Exercises 73–76, sketch the graph of the equation and identify all x- and y-intercepts.

73. $y = -x - 9$

74. $2x + y = 10$

75. $-3x + 8y = 16$

76. $12x - 7y = 22$

In Exercises 77–80, use a graphing utility to graph the exponential function.

77. $f(x) = e^{3x}$

78. $f(x) = -e^{3x}$

79. $f(x) = 2 + e^{3x}$

80. $f(x) = -4 + e^{3x}$

In Exercises 81–84, use a graphing utility to graph the logarithmic function.

81. $f(x) = \log_3 x$

82. $f(x) = \log_3 x + 1$

83. $f(x) = -\log_3 x$

84. $f(x) = \log_3(x - 4)$

4.4 Trigonometric Functions of Any Angle

Introduction

In Section 4.3, the definitions of trigonometric functions were restricted to acute angles. In this section, the definitions are extended to cover *any* angle. If θ is an *acute* angle, the definitions here coincide with those given in the previous section.

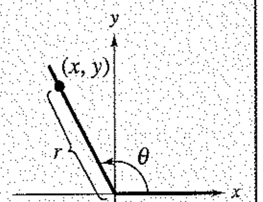
Definitions of Trigonometric Functions of Any Angle

Let θ be an angle in standard position with (x, y) a point on the terminal side of θ and $r = \sqrt{x^2 + y^2} \neq 0$.

$\sin \theta = \dfrac{y}{r}$ $\qquad$ $\cos \theta = \dfrac{x}{r}$

$\tan \theta = \dfrac{y}{x},\quad x \neq 0$ $\qquad$ $\cot \theta = \dfrac{x}{y},\quad y \neq 0$

$\sec \theta = \dfrac{r}{x},\quad x \neq 0$ $\qquad$ $\csc \theta = \dfrac{r}{y},\quad y \neq 0$



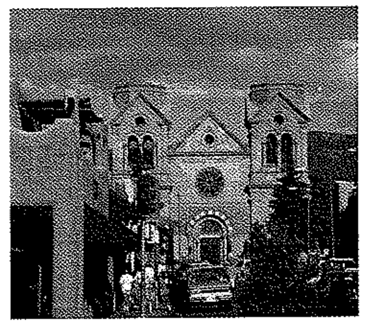
Because $r = \sqrt{x^2 + y^2}$ *cannot* be zero, it follows that the sine and cosine functions are defined for any real value of θ. However, if $x = 0$, the tangent and secant of θ are undefined. For example, the tangent of 90° is undefined. Similarly, if $y = 0$, the cotangent and cosecant of θ are undefined.

What you should learn

- Evaluate trigonometric functions of any angle.
- Use reference angles to evaluate trigonometric functions.
- Evaluate trigonometric functions of real numbers.

Why you should learn it

You can use trigonometric functions to model and solve real-life problems. For instance, Exercise 97 on page 286 shows you how trigonometric functions can be used to model the monthly normal temperature in Santa Fe, New Mexico.



Richard Elliott/Getty Images

Example 1 Evaluating Trigonometric Functions

Let $(-3, 4)$ be a point on the terminal side of θ. Find the sine, cosine, and tangent of θ.

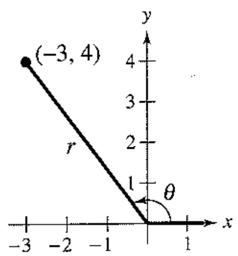


Figure 4.32

Solution

Referring to Figure 4.32, you can see that $x = -3$, $y = 4$, and

$$r = \sqrt{x^2 + y^2} = \sqrt{(-3)^2 + 4^2} = \sqrt{25} = 5.$$

So, you have $\sin \theta = \dfrac{y}{r} = \dfrac{4}{5}$, $\cos \theta = \dfrac{x}{r} = -\dfrac{3}{5}$, and $\tan \theta = \dfrac{y}{x} = -\dfrac{4}{3}$.

✓ *Checkpoint* Now try Exercise 1.

The *signs* of the trigonometric functions in the four quadrants can be determined easily from the definitions of the functions. For instance, because $\cos\theta = x/r$, it follows that $\cos\theta$ is positive wherever $x > 0$, which is in Quadrants I and IV. (Remember, r is always positive.) In a similar manner, you can verify the results shown in Figure 4.33.

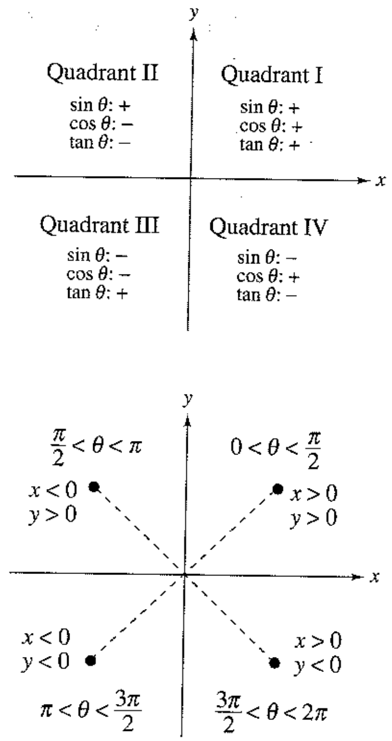


Figure 4.33

Example 2 Evaluating Trigonometric Functions

Given $\tan\theta = -\frac{5}{4}$ and $\cos\theta > 0$, find $\sin\theta$ and $\sec\theta$.

Solution

Note that θ lies in Quadrant IV because that is the only quadrant in which the tangent is negative and the cosine is positive. Moreover, using

$$\tan\theta = \frac{y}{x} = -\frac{5}{4}$$

and the fact that y is negative in Quadrant IV, you can let $y = -5$ and $x = 4$. So, $r = \sqrt{16 + 25} = \sqrt{41}$, and you have the following.

$$\sin\theta = \frac{y}{r} = \frac{-5}{\sqrt{41}} \qquad \text{Exact value}$$

$$\approx -0.7809 \qquad \text{Approximate value}$$

$$\sec\theta = \frac{r}{x} = \frac{\sqrt{41}}{4} \qquad \text{Exact value}$$

$$\approx 1.6008 \qquad \text{Approximate value}$$

✓ ***Checkpoint*** Now try Exercise 19.

Example 3 Trigonometric Functions of Quadrant Angles

Evaluate the sine and cosine functions at the angles $0, \frac{\pi}{2}, \pi$, and $\frac{3\pi}{2}$.

Solution

To begin, choose a point on the terminal side of each angle, as shown in Figure 4.34. For each of the four given points, $r = 1$, and you have the following.

$$\sin 0 = \frac{y}{r} = \frac{0}{1} = 0 \qquad \cos 0 = \frac{x}{r} = \frac{1}{1} = 1 \qquad (x, y) = (1, 0)$$

$$\sin\frac{\pi}{2} = \frac{y}{r} = \frac{1}{1} = 1 \qquad \cos\frac{\pi}{2} = \frac{x}{r} = \frac{0}{1} = 0 \qquad (x, y) = (0, 1)$$

$$\sin\pi = \frac{y}{r} = \frac{0}{1} = 0 \qquad \cos\pi = \frac{x}{r} = \frac{-1}{1} = -1 \qquad (x, y) = (-1, 0)$$

$$\sin\frac{3\pi}{2} = \frac{y}{r} = \frac{-1}{1} = -1 \qquad \cos\frac{3\pi}{2} = \frac{x}{r} = \frac{0}{1} = 0 \qquad (x, y) = (0, -1)$$

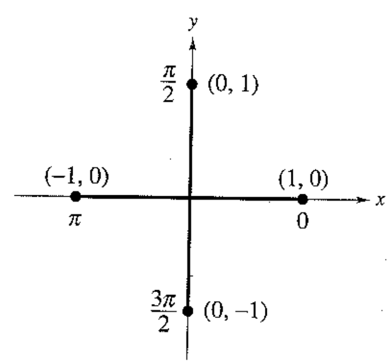


Figure 4.34

✓ ***Checkpoint*** Now try Exercise 29.

Reference Angles

The values of the trigonometric functions of angles greater than 90° (or less than 0°) can be determined from their values at corresponding acute angles called **reference angles.**

> **Definition of Reference Angle**
>
> Let θ be an angle in standard position. Its **reference angle** is the acute angle θ' formed by the terminal side of θ and the horizontal axis.

Figure 4.35 shows the reference angles for θ in Quadrants II, III, and IV.

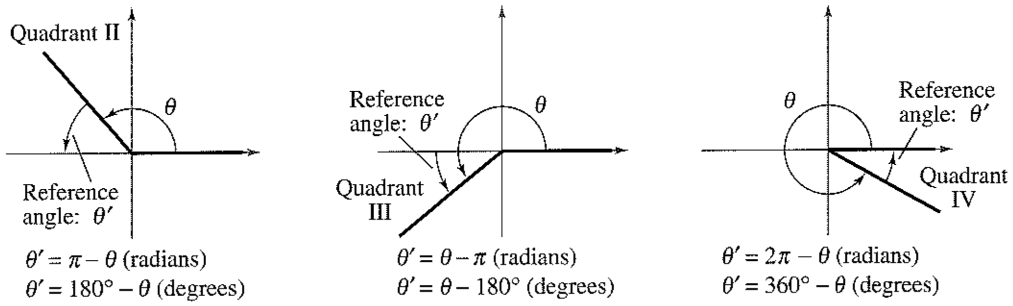


Figure 4.35

Example 4 Finding Reference Angles

Find the reference angle θ'.

a. $\theta = 300°$ **b.** $\theta = 2.3$ **c.** $\theta = -135°$

Solution

a. Because 300° lies in Quadrant IV, the angle it makes with the x-axis is

$$\theta' = 360° - 300° = 60°. \quad \text{Degrees}$$

b. Because 2.3 lies between $\pi/2 \approx 1.5708$ and $\pi \approx 3.1416$, it follows that it is in Quadrant II and its reference angle is

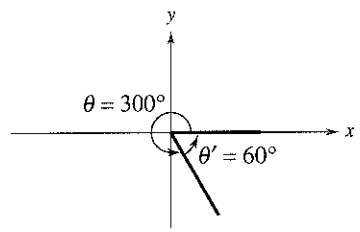
$$\theta' = \pi - 2.3 \approx 0.8416. \quad \text{Radians}$$

c. First, determine that $-135°$ is coterminal with 225°, which lies in Quadrant III. So, the reference angle is

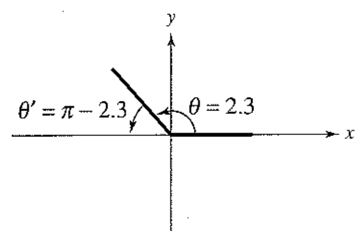
$$\theta' = 225° - 180° = 45°. \quad \text{Degrees}$$

Figure 4.36 shows each angle θ and its reference angle θ'.

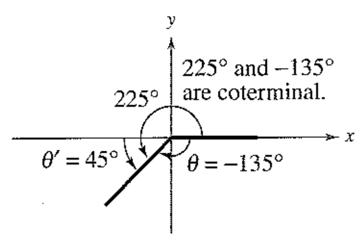
✓ ***Checkpoint*** Now try Exercise 51.



(a)



(b)



(c)

Figure 4.36

Trigonometric Functions of Real Numbers

To see how a reference angle is used to evaluate a trigonometric function, consider the point (x, y) on the terminal side of θ as shown in Figure 4.37. By definition, you know that

$$\sin\theta = \frac{y}{r} \quad \text{and} \quad \tan\theta = \frac{y}{x}.$$

For the right triangle with acute angle θ' and sides of lengths $|x|$ and $|y|$, you have

$$\sin\theta' = \frac{\text{opp}}{\text{hyp}} = \frac{|y|}{r}$$

and

$$\tan\theta' = \frac{\text{opp}}{\text{adj}} = \frac{|y|}{|x|}.$$

So, it follows that $\sin\theta$ and $\sin\theta'$ are equal, *except possibly in sign*. The same is true for $\tan\theta$ and $\tan\theta'$ and for the other four trigonometric functions. In all cases, the sign of the function value can be determined by the quadrant in which θ lies.

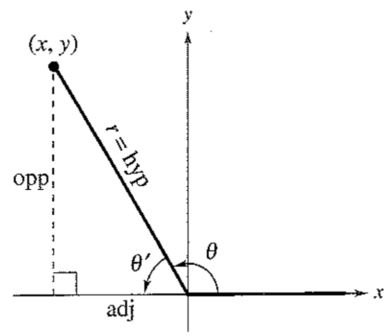


Figure 4.37

Evaluating Trigonometric Functions of Any Angle

To find the value of a trigonometric function of any angle θ:

1. Determine the function value for the associated reference angle θ'.
2. Depending on the quadrant in which θ lies, affix the appropriate sign to the function value.

By using reference angles and the special angles discussed in the previous section, you can greatly extend the scope of *exact* trigonometric values. For instance, knowing the function values of 30° means that you know the function values of all angles for which 30° is a reference angle. For convenience, the following table shows the exact values of the trigonometric functions of special angles and quadrant angles.

Trigonometric Values of Common Angles

θ (degrees)	0°	30°	45°	60°	90°	180°	270°
θ (radians)	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	π	$\frac{3\pi}{2}$
$\sin\theta$	0	$\frac{1}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$	1	0	-1
$\cos\theta$	1	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$	0	-1	0
$\tan\theta$	0	$\frac{\sqrt{3}}{3}$	1	$\sqrt{3}$	Undef.	0	Undef.

STUDY TIP

Learning the table of values at the left is worth the effort because doing so will increase both your efficiency and your confidence. Here is a pattern for the sine function that may help you remember the values.

θ	0°	30°	45°	60°	90°
$\sin\theta$	$\frac{\sqrt{0}}{2}$	$\frac{\sqrt{1}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{4}}{2}$

Reverse the order to get cosine values of the same angles.

Example 5 Trigonometric Functions of Nonacute Angles

Evaluate each trigonometric function.

a. $\cos \dfrac{4\pi}{3}$ **b.** $\tan(-210^\circ)$ **c.** $\csc \dfrac{11\pi}{4}$

Solution

a. Because $\theta = 4\pi/3$ lies in Quadrant III, the reference angle is $\theta' = (4\pi/3) - \pi = \pi/3$, as shown in Figure 4.38. Moreover, the cosine is negative in Quadrant III, so

$$\cos \frac{4\pi}{3} = (-)\cos \frac{\pi}{3} = -\frac{1}{2}.$$

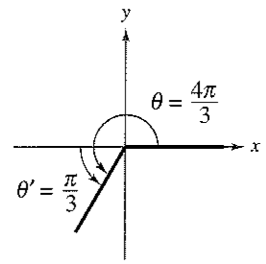


Figure 4.38

b. Because $-210^\circ + 360^\circ = 150^\circ$, it follows that -210° is coterminal with the second-quadrant angle 150°. Therefore, the reference angle is $\theta' = 180^\circ - 150^\circ = 30^\circ$, as shown in Figure 4.39. Finally, because the tangent is negative in Quadrant II, you have

$$\tan(-210^\circ) = (-)\tan 30^\circ = -\frac{\sqrt{3}}{3}.$$

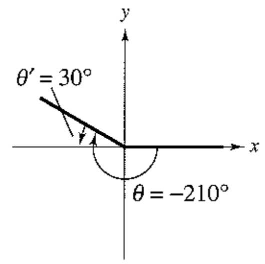


Figure 4.39

c. Because $(11\pi/4) - 2\pi = 3\pi/4$, it follows that $11\pi/4$ is coterminal with the second-quadrant angle $3\pi/4$. Therefore, the reference angle is $\theta' = \pi - (3\pi/4) = \pi/4$, as shown in Figure 4.40. Because the cosecant is positive in Quadrant II, you have

$$\csc \frac{11\pi}{4} = (+)\csc \frac{\pi}{4} = \frac{1}{\sin(\pi/4)} = \sqrt{2}.$$

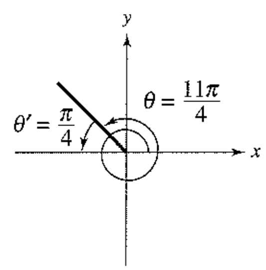


Figure 4.40

✓ ***Checkpoint*** Now try Exercise 65.

The fundamental trigonometric identities listed in the previous section (for an acute angle θ) are also valid when θ is any angle in the domain of the function.

Example 6 Using Trigonometric Identities

Let θ be an angle in Quadrant II such that $\sin \theta = \frac{1}{3}$. Find $\cos \theta$ by using trigonometric identities.

Solution

Using the Pythagorean identity $\sin^2 \theta + \cos^2 \theta = 1$, you obtain

$$\left(\tfrac{1}{3}\right)^2 + \cos^2 \theta = 1$$

$$\cos^2 \theta = 1 - \tfrac{1}{9} = \tfrac{8}{9}.$$

Because $\cos \theta < 0$ in Quadrant II, you can use the negative root to obtain

$$\cos \theta = -\frac{\sqrt{8}}{\sqrt{9}} = -\frac{2\sqrt{2}}{3}.$$

✓ ***Checkpoint*** Now try Exercise 67.

You can use a calculator to evaluate trigonometric functions, as shown in the next example.

Example 7 Using a Calculator

Use a calculator to evaluate each trigonometric function.

a. cot 410° **b.** sin(−7) **c.** sec $\frac{\pi}{9}$

Solution

	Function	*Mode*	*Graphing Calculator Keystrokes*	*Display*
a.	cot 410°	Degree	(TAN 410) x^{-1} ENTER	0.8390996
b.	sin(−7)	Radian	SIN (−) 7) ENTER	−0.6569866
c.	sec $\frac{\pi}{9}$	Radian	(COS (π ÷ 9)) x^{-1} ENTER	1.0641777

Checkpoint Now try Exercise 77.

Exploration

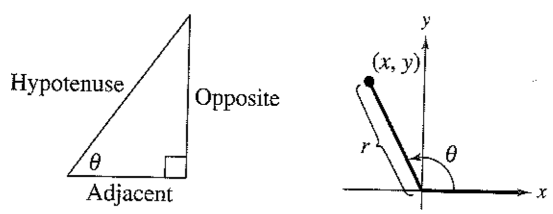
Set your graphing utility to *degree* mode and enter tan 90. What happens? Why? Now set your graphing utility to *radian* mode and enter tan(π/2). Explain the graphing utility's answer.

Library of Functions: Trigonometric Functions

Trigonometric functions are transcendental functions. The six trigonometric functions, sine, cosine, tangent, cosecant, secant, and cotangent, have important uses in construction, surveying, and navigation. Their periodic behavior makes them useful for modeling phenomena such as business cycles, planetary orbits, pendulums, wave motion, and light rays.

The six trigonometric functions can be defined in three different ways.

1. As the ratio of two sides of a right triangle [see Figure 4.41(a)].
2. As coordinates of a point (x, y) in the plane and its distance r from the origin [see Figure 4.41(b)].
3. As functions of any real number, such as time t.



(a) (b)

Figure 4.41

To be efficient in the use of trigonometric functions, you should learn the trigonometric function values of common angles, such as those listed on page 281. Because pairs of trigonometric functions are related to each other by a variety of identities, it is useful to know the fundamental identities presented in Section 4.3. Finally, trigonometric functions and their identity relationships play a prominent role in calculus.

4.4 Exercises

Vocabulary Check

Fill in the blanks. In Exercises 1–6, let θ be an angle in standard position with (x, y) a point on the terminal side of θ and $r = \sqrt{x^2 + y^2} \neq 0$.

1. $\sin \theta =$ ________

2. $\dfrac{r}{y} =$ ________

3. $\tan \theta =$ ________

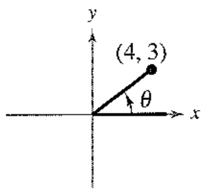
4. $\sec \theta =$ ________

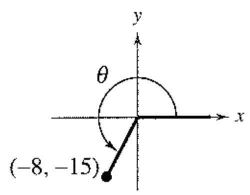
5. $\dfrac{x}{r} =$ ________

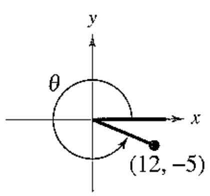
6. $\dfrac{x}{y} =$ ________

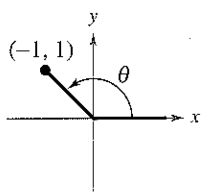
7. The acute positive angle that is formed by the terminal side of the angle θ and the horizontal axis is called the ________ angle of θ and is denoted by θ'.

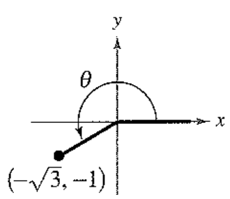
In Exercises 1–4, determine the exact values of the six trigonometric functions of the angle θ.

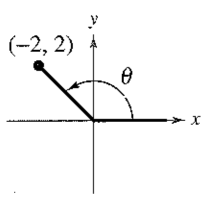
1. (a)


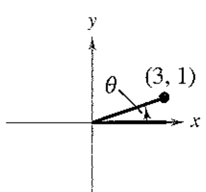
(b)


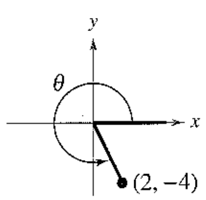
2. (a)


(b)


3. (a)


(b)


4. (a)


(b)


In Exercises 5–12, the point is on the terminal side of an angle in standard position. Determine the exact values of the six trigonometric functions of the angle.

5. $(7, 24)$

6. $(8, 15)$

7. $(5, -12)$

8. $(-24, 10)$

9. $(-4, 10)$

10. $(-5, -6)$

11. $(-10, 8)$

12. $(3, -9)$

In Exercises 13–16, state the quadrant in which θ lies.

13. $\sin \theta < 0$ and $\cos \theta < 0$

14. $\sec \theta > 0$ and $\cot \theta < 0$

15. $\cot \theta > 0$ and $\cos \theta > 0$

16. $\tan \theta > 0$ and $\csc \theta < 0$

In Exercises 17–24, find the values of the six trigonometric functions of θ.

	Function Value	*Constraint*
17.	$\sin \theta = \frac{3}{5}$	θ lies in Quadrant II.
18.	$\cos \theta = -\frac{4}{5}$	θ lies in Quadrant III.
19.	$\tan \theta = -\frac{15}{8}$	$\sin \theta < 0$
20.	$\csc \theta = 4$	$\cot \theta < 0$
21.	$\sec \theta = -2$	$0 \leq \theta \leq \pi$
22.	$\sin \theta = 0$	$\frac{\pi}{2} \leq \theta \leq \frac{3\pi}{2}$
23.	$\cot \theta$ is undefined.	$\frac{\pi}{2} \leq \theta \leq \frac{3\pi}{2}$
24.	$\tan \theta$ is undefined.	$\pi \leq \theta \leq 2\pi$

In Exercises 25–28, the terminal side of θ lies on the given line in the specified quadrant. Find the values of the six trigonometric functions of θ by finding a point on the line.

Line	Quadrant
25. $y = -x$	II
26. $y = \frac{1}{3}x$	III
27. $2x - y = 0$	III
28. $4x + 3y = 0$	IV

In Exercises 29–36, evaluate the trigonometric function of the quadrant angle.

29. $\sec \pi$ **30.** $\tan \frac{\pi}{2}$

31. $\cot \frac{3\pi}{2}$ **32.** $\csc 0$

33. $\sec 0$ **34.** $\csc \frac{3\pi}{2}$

35. $\cot \pi$ **36.** $\csc \frac{\pi}{2}$

In Exercises 37–44, find the reference angle θ' for the special angle θ. Then sketch θ and θ' in standard position.

37. $\theta = 120°$ **38.** $\theta = 225°$

39. $\theta = -135°$ **40.** $\theta = -330°$

41. $\theta = \frac{5\pi}{3}$ **42.** $\theta = \frac{3\pi}{4}$

43. $\theta = -\frac{5\pi}{6}$ **44.** $\theta = -\frac{2\pi}{3}$

In Exercises 45–52, find the reference angle θ' and sketch θ and θ' in standard position.

45. $\theta = 208°$ **46.** $\theta = 322°$

47. $\theta = -292°$ **48.** $\theta = -95°$

49. $\theta = \frac{11\pi}{5}$ **50.** $\theta = \frac{17\pi}{7}$

51. $\theta = -3.68$ **52.** $\theta = -1.72$

In Exercises 53–66, evaluate the sine, cosine, and tangent of the angle without using a calculator.

53. $225°$ **54.** $300°$

55. $-750°$ **56.** $-495°$

57. $-240°$ **58.** $-330°$

59. $\frac{5\pi}{3}$ **60.** $\frac{3\pi}{4}$

61. $-\frac{\pi}{6}$ **62.** $-\frac{4\pi}{3}$

63. $\frac{11\pi}{4}$ **64.** $\frac{10\pi}{3}$

65. $-\frac{17\pi}{6}$ **66.** $-\frac{20\pi}{3}$

In Exercises 67–74, find the indicated trigonometric value in the specified quadrant.

Function	Quadrant	Trigonometric Value
67. $\sin \theta = -\frac{3}{5}$	IV	$\cos \theta$
68. $\cot \theta = -3$	II	$\sin \theta$
69. $\tan \theta = \frac{3}{2}$	III	$\sec \theta$
70. $\csc \theta = -2$	IV	$\cot \theta$
71. $\cos \theta = \frac{5}{8}$	I	$\sec \theta$
72. $\sec \theta = -\frac{9}{4}$	III	$\tan \theta$
73. $\sin \theta = \frac{1}{3}$	II	$\cos \theta$
74. $\tan \theta = -\frac{5}{4}$	IV	$\csc \theta$

In Exercises 75–90, use a calculator to evaluate the trigonometric function. Round you answer to four decimal places. (Be sure the calculator is set in the correct angle mode.)

75. $\sin 10°$ **76.** $\sec 235°$

77. $\tan 245°$ **78.** $\csc 320°$

79. $\cos(-110°)$ **80.** $\cot(-220°)$

81. $\sec(-280°)$ **82.** $\sin(-195°)$

83. $\sin 0.65$ **84.** $\sin(-0.65)$

85. $\cos(-1.81)$ **86.** $\csc 0.33$

87. $\tan \frac{2\pi}{9}$ **88.** $\tan \frac{11\pi}{9}$

89. $\csc\left(-\frac{8\pi}{9}\right)$ **90.** $\cos\left(-\frac{15\pi}{14}\right)$

In Exercises 91–96, find two solutions of the equation. Give your answers in degrees ($0° \le \theta < 360°$) and radians ($0 \le \theta < 2\pi$). Do not use a calculator.

91. (a) $\sin \theta = \frac{1}{2}$ (b) $\sin \theta = -\frac{1}{2}$

92. (a) $\cos \theta = \frac{\sqrt{2}}{2}$ (b) $\cos \theta = -\frac{\sqrt{2}}{2}$

93. (a) $\csc\theta = \frac{2\sqrt{3}}{3}$ (b) $\cot\theta = -1$

94. (a) $\csc\theta = -\sqrt{2}$ (b) $\csc\theta = 2$

95. (a) $\sec\theta = -\frac{2\sqrt{3}}{3}$ (b) $\cos\theta = -\frac{1}{2}$

96. (a) $\cot\theta = -\sqrt{3}$ (b) $\sec\theta = \sqrt{2}$

97. ***Meteorology*** The monthly normal temperature T (in degrees Fahrenheit) for Santa Fe, New Mexico is given by

$$T = 49.5 + 20.5\cos\left(\frac{\pi t}{6} - \frac{7\pi}{6}\right)$$

where t is the time in months, with $t = 1$ corresponding to January. Find the monthly normal temperature for each month. (Source: National Climatic Data Center)

(a) January (b) July (c) December

98. ***Sales*** A company that produces water skis,which are seasonal products, forecasts monthly sales over a two-year period to be

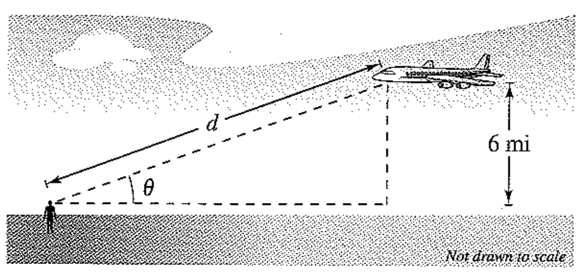
$$S = 23.1 + 0.442t + 4.3\sin\frac{\pi t}{6}$$

where S is measured in thousands of units and t is the time (in months), with $t = 1$ representing January 2004. Estimate sales for each month.

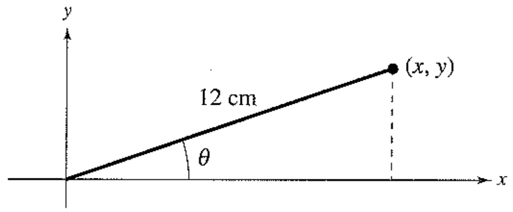
(a) January 2004 (b) February 2005

(c) May 2004 (d) June 2004

99. ***Distance*** An airplane flying at an altitude of 6 miles is on a flight path that passes directly over an observer (see figure). If θ is the angle of elevation from the observer to the plane, find the distance from the observer to the plane when (a) $\theta = 30°$, (b) $\theta = 90°$, and (c) $\theta = 120°$.



100. ***Writing*** Consider an angle in standard position with $r = 12$ centimeters, as shown in the figure. Write a short paragraph describing the change in the magnitudes of x, y, $\sin\theta$, $\cos\theta$, and $\tan\theta$ as θ increases continually from 0° to 90°.



Synthesis

True or False? **In Exercises 101 and 102, determine whether the statement is true or false. Justify your answer.**

101. $\sin 151° = \sin 29°$ **102.** $-\cot\left(\frac{3\pi}{4}\right) = \cot\left(-\frac{\pi}{4}\right)$

103. ***Conjecture***

(a) Use a graphing utility to complete the table.

θ	0°	20°	40°	60°	80°
$\sin\theta$					
$\sin(180° - \theta)$					

(b) Make a conjecture about the relationship between $\sin\theta$ and $\sin(180° - \theta)$.

104. ***Writing*** Create a table of the six trigonometric functions comparing their domains, ranges, evenness, oddness, periods, and zeros. Then identify and write a short paragraph describing any inherent patterns in the trigonometric functions. What can you conclude?

Review

In Exercises 105–114, solve the equation. Round your answer to three decimal places, if necessary.

105. $3x - 7 = 14$ **106.** $44 - 9x = 61$

107. $x^2 - 2x - 5 = 0$ **108.** $2x^2 + x - 4 = 0$

109. $\frac{3}{x-1} = \frac{x+2}{9}$ **110.** $\frac{5}{x} = \frac{x+4}{2x}$

111. $4^{3-x} = 726$ **112.** $\frac{4500}{4 + e^{2x}} = 50$

113. $\ln x = -6$ **114.** $\ln\sqrt{x + 10} = 1$

4.5 Graphs of Sine and Cosine Functions

Basic Sine and Cosine Curves

In this section you will study techniques for sketching the graphs of the sine and cosine functions. The graph of the sine function is a **sine curve.** In Figure 4.42, the black portion of the graph represents one period of the function and is called **one cycle** of the sine curve. The gray portion of the graph indicates that the basic sine wave repeats indefinitely to the right and left. The graph of the cosine function is shown in Figure 4.43. To produce these graphs with a graphing utility, make sure you have set the graphing utility to *radian* mode.

Recall from Section 4.2 that the domain of the sine and cosine functions is the set of all real numbers. Moreover, the range of each function is the interval $[-1, 1]$, and each function has a period of 2π. Do you see how this information is consistent with the basic graphs shown in Figures 4.42 and 4.43?

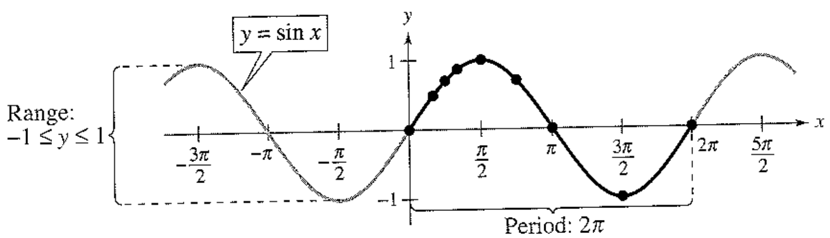


Figure 4.42

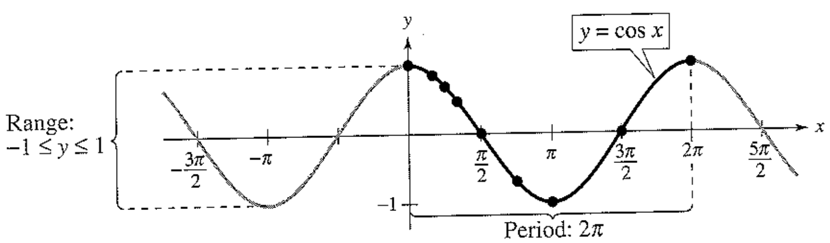


Figure 4.43

The table below lists key points on the graphs of $y = \sin x$ and $y = \cos x$.

x	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{3\pi}{4}$	π	$\frac{3\pi}{2}$	2π
$\sin x$	0	$\frac{1}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$	1	$\frac{\sqrt{2}}{2}$	0	-1	0
$\cos x$	1	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$	0	$-\frac{\sqrt{2}}{2}$	-1	0	1

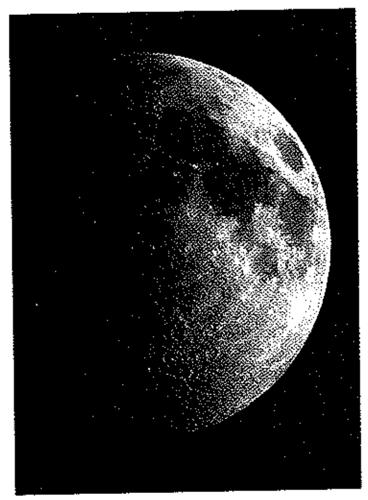
Note from Figures 4.42 and 4.43 that the sine graph is symmetric with respect to the *origin*, whereas the cosine graph is symmetric with respect to the *y-axis*. These properties of symmetry follow from the fact that the sine function is odd whereas the cosine function is even.

What you should learn

- Sketch the graphs of basic sine and cosine functions.
- Use amplitude and period to help sketch the graphs of sine and cosine functions.
- Sketch translations of graphs of sine and cosine functions.
- Use sine and cosine functions to model real-life data.

Why you should learn it

Sine and cosine functions are often used in scientific calculations. For instance, in Exercise 79 on page 296, you can use a trigonometric function to model the percent of the moon's face that is illuminated for any given day in 2006.



Jerry Lodriguss/Photo Researchers, Inc.

To sketch the graphs of the basic sine and cosine functions by hand, it helps to note five *key points* in one period of each graph: the *intercepts*, the *maximum points*, and the *minimum points*. See Figure 4.44.

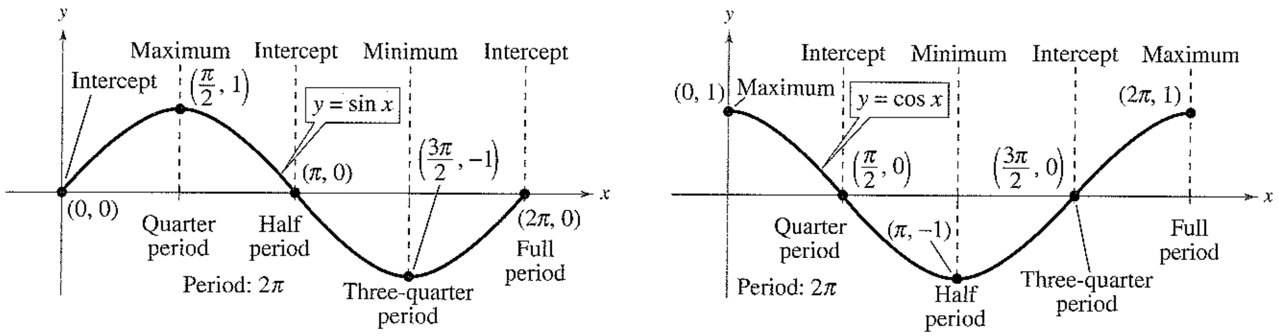


Figure 4.44

Example 1 Using Key Points to Sketch a Sine Curve

Sketch the graph of $y = 2 \sin x$ by hand on the interval $[-\pi, 4\pi]$.

Solution

Note that

$$y = 2 \sin x = 2(\sin x)$$

indicates that the y-values of the key points will have twice the magnitude of those on the graph of $y = \sin x$. Divide the period 2π into four equal parts to get the key points

Intercept	*Maximum*	*Intercept*	*Minimum*		*Intercept*
$(0, 0)$,	$\left(\frac{\pi}{2}, 2\right)$,	$(\pi, 0)$,	$\left(\frac{3\pi}{2}, -2\right)$,	and	$(2\pi, 0)$.

By connecting these key points with a smooth curve and extending the curve in both directions over the interval $[-\pi, 4\pi]$, you obtain the graph shown in Figure 4.45. Use a graphing utility to confirm this graph. Be sure to set the graphing utility to *radian* mode.

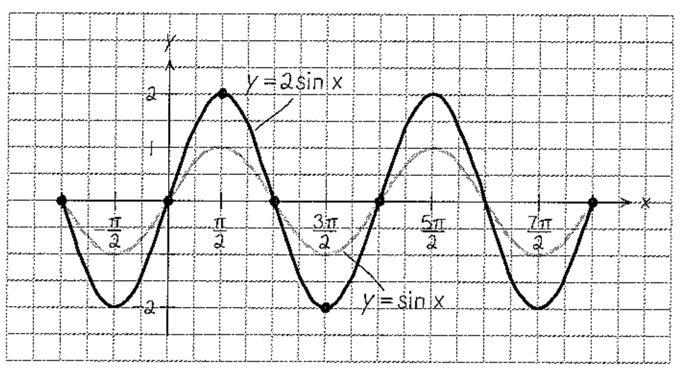
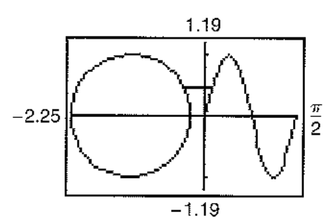


Figure 4.45

Checkpoint Now try Exercise 39.

Exploration

Enter the Graphing a Sine Function Program found on our website, *college.hmco.com*, into your graphing utility. This program simultaneously draws the unit circle and the corresponding points on the sine curve, as shown below. After the circle and sine curve are drawn, you can connect the points on the unit circle with their corresponding points on the sine curve by pressing ENTER. Discuss the relationship that is illustrated.



Amplitude and Period of Sine and Cosine Curves

In the rest of this section you will study the graphic effect of each of the constants a, b, c, and d in equations of the forms

$$y = d + a\sin(bx - c) \qquad \text{and} \qquad y = d + a\cos(bx - c).$$

A quick review of the transformations studied in Section 1.5 should help in this investigation.

The constant factor a in $y = a\sin x$ acts as a *scaling factor*—a *vertical stretch* or *vertical shrink* of the basic sine curve. If $|a| > 1$, the basic sine curve is stretched, and if $|a| < 1$, the basic sine curve is shrunk. The result is that the graph of $y = a\sin x$ ranges between $-a$ and a instead of between -1 and 1. The absolute value of a is the **amplitude** of the function $y = a\sin x$. The range of the function $y = a\sin x$ for $a > 0$ is $-a \le y \le a$.

Definition of Amplitude of Sine and Cosine Curves

The **amplitude** of $y = a\sin x$ and $y = a\cos x$ represents half the distance between the maximum and minimum values of the function and is given by

Amplitude $= |a|$.

TECHNOLOGY TIP

When using a graphing utility to graph trigonometric functions, pay special attention to the viewing window you use. For instance, try graphing $y = [\sin(10x)]/10$ in the standard viewing window in *radian* mode. What do you observe? Use the *zoom* feature to find a viewing window that displays a good view of the graph. For instructions on how to use the *zoom* feature, see Appendix A; for specific keystrokes, go to the text website at *college.hmco.com*.

Example 2 Scaling: Vertical Shrinking and Stretching

On the same set of coordinate axes, sketch the graph of each function by hand.

a. $y = \frac{1}{2}\cos x$ **b.** $y = 3\cos x$

Solution

a. Because the amplitude of $y = \frac{1}{2}\cos x$ is $\frac{1}{2}$, the maximum value is $\frac{1}{2}$ and the minimum value is $-\frac{1}{2}$. Divide one cycle, $0 \le x \le 2\pi$, into four equal parts to get the key points

Maximum	*Intercept*	*Minimum*	*Intercept*		*Maximum*
$\left(0, \frac{1}{2}\right)$,	$\left(\frac{\pi}{2}, 0\right)$,	$\left(\pi, -\frac{1}{2}\right)$,	$\left(\frac{3\pi}{2}, 0\right)$,	and	$\left(2\pi, \frac{1}{2}\right)$.

b. A similar analysis shows that the amplitude of $y = 3\cos x$ is 3, and the key points are

Maximum	*Intercept*	*Minimum*	*Intercept*		*Maximum*
$(0, 3)$,	$\left(\frac{\pi}{2}, 0\right)$,	$(\pi, -3)$,	$\left(\frac{3\pi}{2}, 0\right)$,	and	$(2\pi, 3)$.

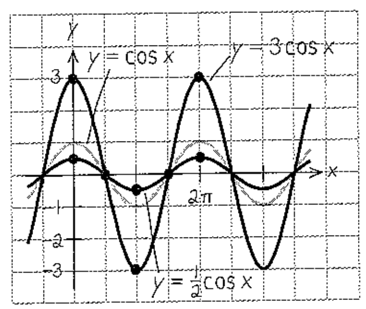


Figure 4.46

The graphs of these two functions are shown in Figure 4.46. Notice that the graph of $y = \frac{1}{2}\cos x$ is a vertical shrink of the graph of $y = \cos x$ and the graph of $y = 3\cos x$ is a vertical stretch of the graph of $y = \cos x$. Use a graphing utility to confirm these graphs.

Checkpoint Now try Exercise 40.

You know from Section 1.4 that the graph of $y = -f(x)$ is a *reflection* in the x-axis of the graph of $y = f(x)$. For instance, the graph of $y = -3 \cos x$ is a reflection of the graph of $y = 3 \cos x$, as shown in Figure 4.47.

Because $y = a \sin x$ completes one cycle from $x = 0$ to $x = 2\pi$, it follows that $y = a \sin bx$ completes one cycle from $x = 0$ to $x = 2\pi/b$.

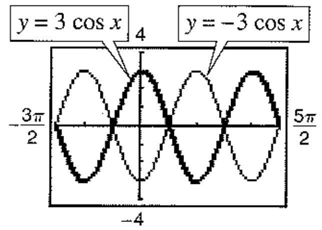


Figure 4.47

Period of Sine and Cosine Functions

Let b be a positive real number. The **period** of $y = a \sin bx$ and $y = a \cos bx$ is given by

$$\text{Period} = \frac{2\pi}{b}.$$

Note that if $0 < b < 1$, the period of $y = a \sin bx$ is greater than 2π and represents a *horizontal stretching* of the graph of $y = a \sin x$. Similarly, if $b < 1$, the period of $y = a \sin bx$ is less than 2π and represents a *horizontal shrinking* of the graph of $y = a \sin x$. If b is negative, the identities $\sin(-x) = -\sin x$ and $\cos(-x) = \cos x$ are used to rewrite the function.

Example 3 Scaling: Horizontal Stretching

Sketch the graph of $y = \sin \frac{x}{2}$ by hand.

Solution

The amplitude is 1. Moreover, because $b = \frac{1}{2}$, the period is

$$\frac{2\pi}{b} = \frac{2\pi}{\frac{1}{2}} = 4\pi. \qquad \text{Substitute for } b.$$

Now, divide the period-interval $[0, 4\pi]$ into four equal parts with the values π, 2π, and 3π to obtain the key points on the graph

Intercept	*Maximum*	*Intercept*	*Minimum*		*Intercept*
$(0, 0)$,	$(\pi, 1)$,	$(2\pi, 0)$,	$(3\pi, -1)$,	and	$(4\pi, 0)$.

The graph is shown in Figure 4.48. Use a graphing utility to confirm this graph.

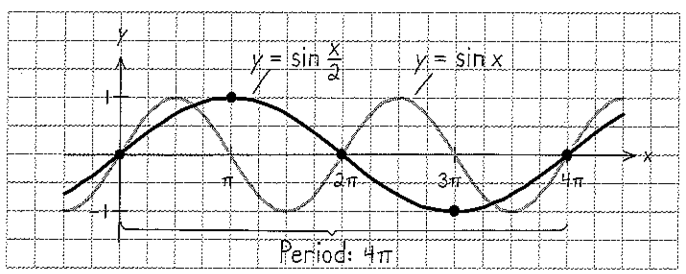


Figure 4.48

✓ *Checkpoint* Now try Exercise 41.

STUDY TIP

In general, to divide a period-interval into four equal parts, successively add "period/4," starting with the left endpoint of the interval. For instance, for the period-interval $[-\pi/6, \pi/2]$ of length $2\pi/3$, you would successively add

$$\frac{2\pi/3}{4} = \frac{\pi}{6}$$

to get $-\pi/6, 0, \pi/6, \pi/3$, and $\pi/2$ as the key points on the graph.

Translations of Sine and Cosine Curves

The constant c in the general equations

$$y = a\sin(bx - c) \quad \text{and} \quad y = a\cos(bx - c)$$

creates *horizontal translations* (shifts) of the basic sine and cosine curves. Comparing $y = a\sin bx$ with $y = a\sin(bx - c)$, you find that the graph of $y = a\sin(bx - c)$ completes one cycle from $bx - c = 0$ to $bx - c = 2\pi$. By solving for x, you can find the interval for one cycle to be

$$\underbrace{\frac{c}{b}}_{\text{Left endpoint}} \le x \le \underbrace{\frac{c}{b} + \underbrace{\frac{2\pi}{b}}_{\text{Period}}}_{\text{Right endpoint}}.$$

This implies that the period of $y = a\sin(bx - c)$ is $2\pi/b$, and the graph of $y = a\sin bx$ is shifted by an amount c/b. The number c/b is the **phase shift.**

> **Graphs of Sine and Cosine Functions**
>
> The graphs of $y = a\sin(bx - c)$ and $y = a\cos(bx - c)$ have the following characteristics. (Assume $b > 0$.)
>
> Amplitude $= |a|$ Period $= 2\pi/b$
>
> The left and right endpoints of a one-cycle interval can be determined by solving the equations $bx - c = 0$ and $bx - c = 2\pi$.

TECHNOLOGY SUPPORT

For instructions on how to use the *minimum* feature, the *maximum* feature, and the *zero* or *root* feature, see Appendix A; for specific keystrokes, go to the text website at *college.hmco.com*.

Example 4 Horizontal Translation

Analyze the graph of $y = \frac{1}{2}\sin(x - \pi/3)$.

Algebraic Solution

The amplitude is $\frac{1}{2}$ and the period is 2π. By solving the equations

$$x - \frac{\pi}{3} = 0 \quad \text{and} \quad x - \frac{\pi}{3} = 2\pi$$

$$x = \frac{\pi}{3} \qquad\qquad x = \frac{7\pi}{3}$$

you see that the interval $[\pi/3, 7\pi/3]$ corresponds to one cycle of the graph. Dividing this interval into four equal parts produces the following key points.

Intercept	*Maximum*	*Intercept*	*Minimum*	*Intercept*
$\left(\frac{\pi}{3}, 0\right)$,	$\left(\frac{5\pi}{6}, \frac{1}{2}\right)$,	$\left(\frac{4\pi}{3}, 0\right)$,	$\left(\frac{11\pi}{6}, -\frac{1}{2}\right)$,	$\left(\frac{7\pi}{3}, 0\right)$

✓ *Checkpoint* Now try Exercise 45.

Graphical Solution

Use a graphing utility set in *radian* mode to graph $y = (1/2)\sin(x - \pi/3)$, as shown in Figure 4.49. Use the *minimum*, *maximum*, and *zero* or *root* features of the graphing utility to approximate the key points $(1.047, 0)$, $(2.618, 0.5)$, $(4.189, 0)$, $(5.760, -0.5)$, and $(7.330, 0)$.

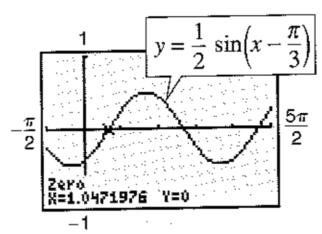


Figure 4.49

Example 5 Horizontal Translation

Use a graphing utility to analyze the graph of $y = -3\cos(2\pi x + 4\pi)$.

Solution

The amplitude is 3 and the period is $2\pi/2\pi = 1$. By solving the equations

$$2\pi x + 4\pi = 0 \qquad \text{and} \qquad 2\pi x + 4\pi = 2\pi$$
$$2\pi x = -4\pi \qquad\qquad 2\pi x = -2\pi$$
$$x = -2 \qquad\qquad x = -1$$

you see that the interval $[-2, -1]$ corresponds to one cycle of the graph. Dividing this interval into four equal parts produces the key points

Minimum	*Intercept*	*Maximum*	*Intercept*		*Minimum*
$(-2, -3)$,	$(-7/4, 0)$,	$(-3/2, 3)$,	$(-5/4, 0)$,	and	$(-1, -3)$.

The graph is shown in Figure 4.50.

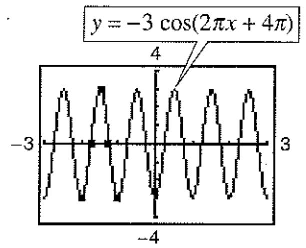


Figure 4.50

✓ *Checkpoint* Now try Exercise 47.

The final type of transformation is the *vertical translation* caused by the constant d in the equations

$$y = d + a\sin(bx - c) \qquad \text{and} \qquad y = d + a\cos(bx - c).$$

The shift is d units upward for $d > 0$ and d units downward for $d < 0$. In other words, the graph oscillates about the horizontal line $y = d$ instead of about the x-axis.

Example 6 Vertical Translation

Use a graphing utility to analyze the graph of $y = 2 + 3\cos 2x$.

Solution

The amplitude is 3 and the period is π. The key points over the interval $[0, \pi]$ are

$$(0, 5), \quad (\pi/4, 2), \quad (\pi/2, -1), \quad (3\pi/4, 2), \quad \text{and} \quad (\pi, 5).$$

The graph is shown in Figure 4.51. Compared with the graph of $f(x) = 3\cos 2x$, the graph of $y = 2 + 3\cos 2x$ is shifted upward two units.

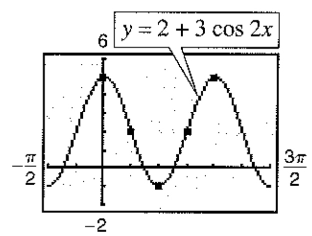


Figure 4.51

✓ *Checkpoint* Now try Exercise 49.

Example 7 Finding an Equation for a Graph

Find the amplitude, period, and phase shift for the sine function whose graph is shown in Figure 4.52. Write an equation for this graph.

Solution

The amplitude of this sine curve is 2. The period is 2π, and there is a right phase shift of $\pi/2$. So, you can write $y = 2\sin(x - \pi/2)$.

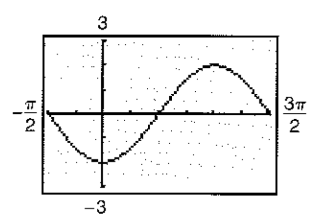


Figure 4.52

✓ *Checkpoint* Now try Exercise 67.

Mathematical Modeling

Sine and cosine functions can be used to model many real-life situations, including electric currents, musical tones, radio waves, tides, and weather patterns.

Example 8 Finding a Trigonometric Model

Throughout the day, the depth of the water at the end of a dock in Bangor, Washington varies with the tides. The table shows the depths (in feet) at various times during the morning. (Source: Nautical Software, Inc.)

Time	Depth, y
Midnight	3.1
2 A.M.	7.8
4 A.M.	11.3
6 A.M.	10.9
8 A.M.	6.6
10 A.M.	1.7
Noon	0.9

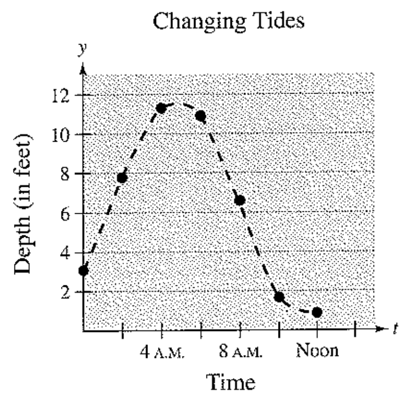


Figure 4.53

a. Use a trigonometric function to model this data.

b. A boat needs at least 10 feet of water to moor at the dock. During what times in the evening can it safely dock?

Solution

a. Begin by graphing the data, as shown in Figure 4.53. You can use either a sine or cosine model. Suppose you use a cosine model of the form

$$y = a\cos(bt - c) + d.$$

The difference between the maximum height and minimum height of the graph is twice the amplitude of the function. So, the amplitude is

$$a = \tfrac{1}{2}[(\text{maximum depth}) - (\text{minimum depth})] = \tfrac{1}{2}(11.3 - 0.9) = 5.2.$$

The cosine function completes one half of a cycle between the times at which the maximum and minimum depths occur. So, the period is

$$p = 2[(\text{time of min. depth}) - (\text{time of max. depth})] = 2(12 - 4) = 16$$

which implies that $b = 2\pi/p \approx 0.393$. Because high tide occurs 4 hours after midnight, consider the left endpoint to be $c/b = 4$, so $c \approx 1.571$. Moreover, because the average depth is $\frac{1}{2}(11.3 + 0.9) = 6.1$, it follows that $d = 6.1$. So, you can model the depth with the function

$$y = 5.2\cos(0.393t - 1.571) + 6.1.$$

b. Using a graphing utility, graph the model with the line $y = 10$. Using the *intersect* feature, you can determine that the depth is at least 10 feet between 6:06 P.M. ($t \approx 18.1$) and 9:48 P.M. ($t \approx 21.8$), as shown in Figure 4.54.

TECHNOLOGY SUPPORT

For instructions on how to use the *intersect* feature, see Appendix A; for specific keystrokes, go to the text website at *college.hmco.com*.

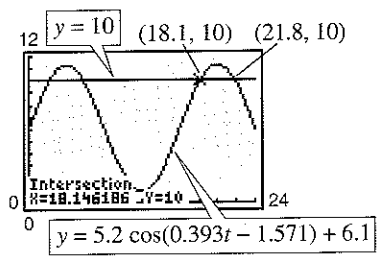


Figure 4.54

Checkpoint Now try Exercise 79.

4.5 Exercises

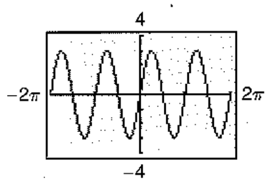
Vocabulary Check

Fill in the blanks.

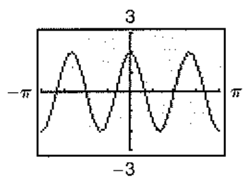
1. The ________ of a sine or cosine curve represents half the distance between the maximum and minimum values of the function.
2. One period of a sine function is called ________ of the sine curve.
3. The period of a sine or cosine function is given by ________ .
4. For the equation $y = a\sin(bx - c)$, $\frac{c}{b}$ is the ________ of the graph of the equation.

In Exercises 1–14, find the period and amplitude.

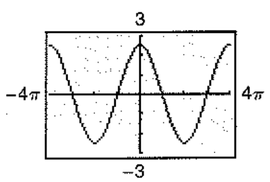
1. $y = 3\sin 2x$



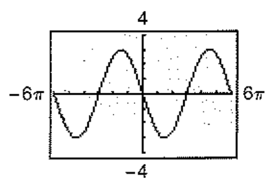
2. $y = 2\cos 3x$



3. $y = \frac{5}{2}\cos\frac{x}{2}$



4. $y = -3\sin\frac{x}{3}$



5. $y = \frac{2}{3}\sin \pi x$

6. $y = \frac{3}{2}\cos\frac{\pi x}{2}$

7. $y = -2\sin x$

8. $y = -\cos\frac{2x}{5}$

9. $y = 3\sin 10x$

10. $y = \frac{1}{3}\sin 10x$

11. $y = \frac{1}{4}\cos\frac{2x}{3}$

12. $y = \frac{5}{2}\cos\frac{x}{4}$

13. $y = \frac{1}{3}\sin 4\pi x$

14. $y = \frac{3}{4}\cos\frac{\pi x}{12}$

In Exercises 15–22, describe the relationship between the graphs of *f* and *g*. Consider amplitudes, periods, and shifts.

15. $f(x) = \sin x$
$g(x) = \sin(x - \pi)$

16. $f(x) = \cos x$
$g(x) = \cos(x + \pi)$

17. $f(x) = \cos 2x$
$g(x) = -\cos 2x$

18. $f(x) = \sin 3x$
$g(x) = \sin(-3x)$

19. $f(x) = \cos x$
$g(x) = -5\cos x$

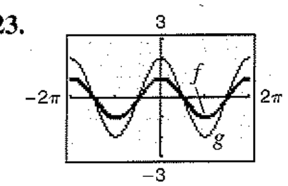
20. $f(x) = \sin x$
$g(x) = -\frac{1}{2}\sin x$

21. $f(x) = \sin 2x$
$g(x) = 5 + \sin 2x$

22. $f(x) = \cos 4x$
$g(x) = -6 + \cos 4x$

In Exercises 23–26, describe the relationship between the graphs of *f* and *g*. Consider amplitudes, periods, and shifts.

23.



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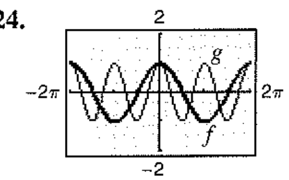

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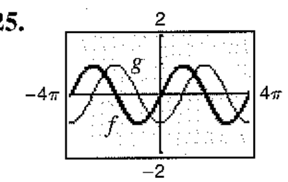

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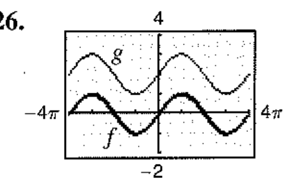


In Exercises 27–34, sketch the graphs of *f* and *g* in the same coordinate plane. (Include two full periods.)

27. $f(x) = \sin x$
$g(x) = -4\sin x$

28. $f(x) = \sin x$
$g(x) = \sin\frac{x}{3}$

29. $f(x) = \cos x$
$g(x) = 4 + \cos x$

30. $f(x) = 2\cos 2x$
$g(x) = -\cos 4x$

31. $f(x) = -\frac{1}{2}\sin\frac{x}{2}$

$g(x) = 3 - \frac{1}{2}\sin\frac{x}{2}$

32. $f(x) = 4\sin\pi x$

$g(x) = 4\sin\pi x - 2$

33. $f(x) = 2\cos x$

$g(x) = 2\cos(x + \pi)$

34. $f(x) = -\cos x$

$g(x) = -\cos\left(x - \frac{\pi}{2}\right)$

Conjecture **In Exercises 35–38, use a graphing utility to graph f and g in the same viewing window. (Include two full periods.) Make a conjecture about the functions.**

35. $f(x) = \sin x$

$g(x) = \cos\left(x - \frac{\pi}{2}\right)$

36. $f(x) = \sin x$

$g(x) = -\cos\left(x + \frac{\pi}{2}\right)$

37. $f(x) = \cos x$

$g(x) = -\sin\left(x - \frac{\pi}{2}\right)$

38. $f(x) = \cos x$

$g(x) = -\cos(x - \pi)$

In Exercises 39–54, sketch the graph of the function by hand. Use a graphing utility to verify your sketch. (Include two full periods.)

39. $y = 3\sin x$

40. $y = \frac{1}{4}\cos x$

41. $y = \cos\frac{x}{2}$

42. $y = \sin 4x$

43. $y = -2\sin\frac{2\pi x}{3}$

44. $y = -10\cos\frac{\pi x}{6}$

45. $y = \sin\left(x - \frac{\pi}{4}\right)$

46. $y = \sin(x - \pi)$

47. $y = -8\cos(x + \pi)$

48. $y = 6\cos\left(x + \frac{\pi}{3}\right)$

49. $y = 1 + \frac{1}{2}\cos 4\pi t$

50. $y = -4 + 5\cos\frac{\pi t}{12}$

51. $y = 2 - 2\sin\frac{2\pi x}{3}$

52. $y = 2\cos x - 3$

53. $y = \frac{2}{3}\cos\left(\frac{x}{2} - \frac{\pi}{4}\right)$

54. $y = -3\cos(6x + \pi)$

In Exercises 55–62, use a graphing utility to graph the function. (Include two full periods.) Identify the amplitude and period of the graph.

55. $y = -2\sin(4x + \pi)$

56. $y = -4\sin\left(\frac{2}{3}x - \frac{\pi}{3}\right)$

57. $y = \cos\left(2\pi x - \frac{\pi}{2}\right) + 1$

58. $y = 3\cos\left(\frac{\pi x}{2} + \frac{\pi}{2}\right) - 3$

59. $y = 5\sin(\pi - 2x) + 10$

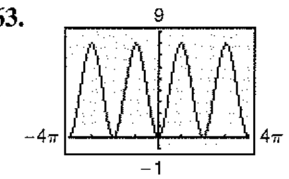
60. $y = 5\cos(\pi - 2x) + 6$

61. $y = \frac{1}{100}\sin 120\pi t$

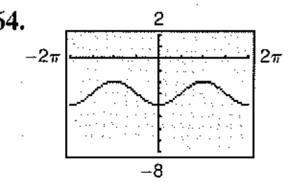
62. $y = -\frac{1}{100}\cos 50\pi t$

Graphical Reasoning **In Exercises 63–66, find a and d for the function $f(x) = a\cos x + d$ such that the graph of f matches the figure.**

63.



64.



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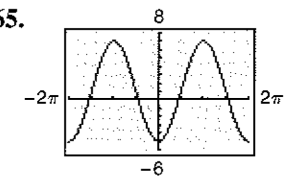

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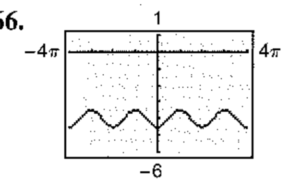
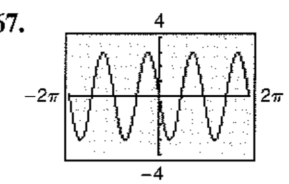
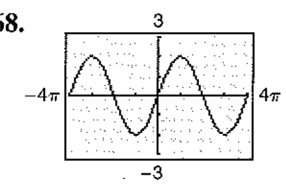


Graphical Reasoning **In Exercises 67–70, find a, b, and c for the function $f(x) = a\sin(bx - c)$ such that the graph of f matches the graph shown.**

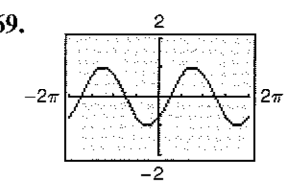
67.



68.



69.



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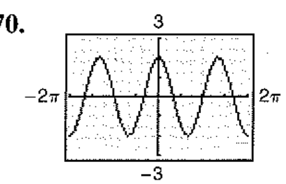


In Exercises 71 and 72, use a graphing utility to graph y_1 and y_2 for all real numbers x in the interval $[-2\pi, 2\pi]$. Use the graphs to find the real numbers x such that $y_1 = y_2$.

71. $y_1 = \sin x$

$y_2 = -\frac{1}{2}$

72. $y_1 = \cos x$

$y_2 = -1$

73. ***Health*** For a person at rest, the velocity v (in liters per second) of air flow during a respiratory cycle (the time from the beginning of one breath to the beginning of the next) is given by

$$v = 0.85 \sin \frac{\pi t}{3}$$

where t is the time (in seconds). (Inhalation occurs when $v > 0$, and exhalation occurs when $v < 0$.)

(a) Use a graphing utility to graph v.

(b) Find the time for one full respiratory cycle.

(c) Find the number of cycles per minute.

(d) The model is for a person at rest. How might the model change for a person who is exercising? Explain.

74. ***Sales*** A company that produces snowboards, which are seasonal products, forecasts monthly sales for 1 year to be

$$S = 74.50 + 43.75 \cos \frac{\pi t}{6}$$

where S is the sales in thousands of units and t is the time in months, with $t = 1$ corresponding to January.

(a) Use a graphing utility to graph the sales function over the one-year period.

(b) Use the graph in part (a) to determine the months of maximum and minimum sales.

75. ***Recreation*** You are riding a Ferris wheel. Your height h (in feet) above the ground at any time t (in seconds) can be modeled by

$$h = 25 \sin \frac{\pi}{15}(t - 75) + 30.$$

The Ferris wheel turns for 135 seconds before it stops to let the first passengers off.

(a) Graph the model.

(b) What are the minimum and maximum heights above the ground?

76. ***Health*** The pressure P (in millimeters of mercury) against the walls of the blood vessels of a person is modeled by

$$P = 100 - 20 \cos \frac{8\pi}{3}t$$

where t is the time (in seconds). Graph the model. One cycle is equivalent to one heartbeat. What is the person's pulse rate in heartbeats per minute?

77. ***Fuel Consumption*** The daily consumption C (in gallons) of diesel fuel on a farm is modeled by

$$C = 30.3 + 21.6 \sin\left(\frac{2\pi t}{365} + 10.9\right)$$

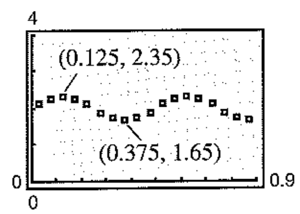
where t is the time in days, with $t = 1$ corresponding to January 1.

(a) What is the period of the model? Is it what you expected? Explain.

(b) What is the average daily fuel consumption? Which term of the model did you use? Explain.

(c) Use a graphing utility to graph the model. Use the graph to approximate the time of the year when consumption exceeds 40 gallons per day.

78. ***Data Analysis*** The motion of an oscillating weight suspended from a spring was measured by a motion detector. The data was collected, and the approximate maximum displacements from equilibrium ($y = 2$) are labeled in the figure. The distance y from the motion detector is measured in centimeters and the time t is measured in seconds.



(a) Is y a function of t? Explain.

(b) Approximate the amplitude and period.

(c) Find a model for the data.

(d) Use a graphing utility to graph the model in part (c). Compare the result with the data in the figure.

79. ***Data Analysis*** The percent y of the moon's face that is illuminated on day x of the year 2006, where $x = 1$ represents January 1, is shown in the table. (Source: U.S. Naval Observatory)

Day, x	Percent, y
29	0.0
36	0.5
44	1.0
52	0.5
58	0.0
65	0.5

(a) Create a scatter plot of the data.

(b) Find a trigonometric model for the data.

(c) Add the graph of your model in part (b) to the scatter plot. How well does the model fit the data?

(d) What is the period of the model?

(e) Estimate the percent illumination of the moon on June 22, 2007.

80. ***Data Analysis*** The table shows the average daily high temperatures for Nantucket, Massachusetts N and Athens, Georgia A (in degrees Fahrenheit) for month t, with $t = 1$ corresponding to January. (Source: U.S. Weather Bureau and the National Weather Service)

Month, t	Nantucket, N	Athens, A
1	40	52
2	41	56
3	42	65
4	53	73
5	62	81
6	71	87
7	78	90
8	76	88
9	70	83
10	59	74
11	48	64
12	40	55

(a) A model for the temperature in Nantucket is given by

$$N(t) = 58 + 19\sin\left(\frac{2\pi t}{11} - \frac{21\pi}{25}\right).$$

Find a trigonometric model for Athens.

(b) Use a graphing utility to graph the data and the model for the temperatures in Nantucket in the same viewing window. How well does the model fit the data?

(c) Use a graphing utility to graph the data and the model for the temperatures in Athens in the same viewing window. How well does the model fit the data?

(d) Use the models to estimate the average daily high temperature in each city. Which term of the models did you use? Explain.

(e) What is the period of each model? Are the periods what you expected? Explain.

(f) Which city has the greater variability in temperature throughout the year? Which factor of the models determines this variability? Explain.

Synthesis

True or False? **In Exercises 81–83, determine whether the statement is true or false. Justify your answer.**

81. The graph of $y = 6 - \frac{3}{4}\sin\frac{3x}{10}$ has a period of $\frac{20\pi}{3}$.

82. The function $y = \frac{1}{2}\cos 2x$ has an amplitude that is twice that of the function $y = \cos x$.

83. The graph of $y = -\cos x$ is a reflection of the graph of $y = \sin(x + \pi/2)$ in the x-axis.

84. ***Writing*** Use a graphing utility to graph the function

$$y = d + a\sin(bx - c)$$

for different values of a, b, c, and d. Write a paragraph describing the changes in the graph corresponding to changes in each variable.

85. ***Exploration*** In Section 4.2 it was shown that $f(x) = \cos x$ is an even function and $g(x) = \sin x$ is an odd function. Use a graphing utility to graph h and use the graph to determine whether h is even, odd, or neither.

(a) $h(x) = \cos^2 x$ (b) $h(x) = \sin^2 x$

(c) $h(x) = \sin x \cos x$

86. ***Conjecture*** If f is an even function and g is an odd function, use the results of Exercise 85 to make a conjecture about each of the following.

(a) $h(x) = [f(x)]^2$ (b) $h(x) = [g(x)]^2$

(c) $h(x) = f(x)g(x)$

Review

In Exercises 87 and 88, plot the points and find the slope of the line passing through the points.

87. $(0, 1), (2, 7)$ **88.** $(-1, 4), (3, -2)$

In Exercises 89 and 90, convert the angle measure from radians to degrees. Round your answer to three decimal places.

89. 8.5 **90.** -0.48

4.6 Graphs of Other Trigonometric Functions

Graph of the Tangent Function

Recall that the tangent function is odd. That is, $\tan(-x) = -\tan x$. Consequently, the graph of $y = \tan x$ is symmetric with respect to the origin. You also know from the identity $\tan x = \sin x/\cos x$ that the tangent function is undefined at values at which $\cos x = 0$. Two such values are $x = \pm\pi/2 \approx \pm 1.5708$.

x	$-\frac{\pi}{2}$	-1.57	-1.5	$-\frac{\pi}{4}$	0	$\frac{\pi}{4}$	1.5	1.57	$\frac{\pi}{2}$
$\tan x$	Undef.	-1255.8	-14.1	-1	0	1	14.1	1255.8	Undef.

tan x approaches $-\infty$ as x approaches $-\pi/2$ from the right.

tan x approaches ∞ as x approaches $\pi/2$ from the left.

As indicated in the table, $\tan x$ increases without bound as x approaches $\pi/2$ from the left, and it decreases without bound as x approaches $-\pi/2$ from the right. So, the graph of $y = \tan x$ has *vertical asymptotes* at $x = \pi/2$ and $-\pi/2$, as shown in Figure 4.55. Moreover, because the period of the tangent function is π, vertical asymptotes also occur at $x = \pi/2 + n\pi$, where n is an integer. The domain of the tangent function is the set of all real numbers other than $x = \pi/2 + n\pi$, and the range is the set of all real numbers.

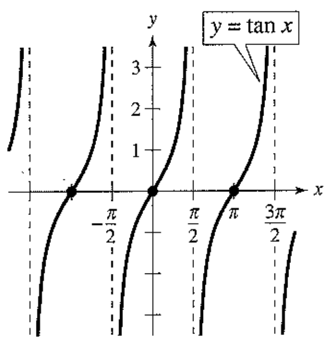


Figure 4.55

Period: π

Domain: all $x \neq \frac{\pi}{2} + n\pi$

Range: $(-\infty, \infty)$

Vertical asymptotes: $x = \frac{\pi}{2} + n\pi$

Sketching the graph of $y = a\tan(bx - c)$ is similar to sketching the graph of $y = a\sin(bx - c)$ in that you locate key points that identify the intercepts and asymptotes. Two consecutive asymptotes can be found by solving the equations $bx - c = -\pi/2$ and $bx - c = \pi/2$. The midpoint between two consecutive asymptotes is an x-intercept of the graph. The period of the function $y = a\tan(bx - c)$ is the distance between two consecutive asymptotes. The amplitude of a tangent function is not defined. After plotting the asymptotes and the x-intercept, plot a few additional points between the two asymptotes and sketch one cycle. Finally, sketch one or two additional cycles to the left and right.

What you should learn

- Sketch the graphs of tangent functions.
- Sketch the graphs of cotangent functions.
- Sketch the graphs of secant and cosecant functions.
- Sketch the graphs of damped trigonometric functions.

Why you should learn it

You can use tangent, cotangent, secant, and cosecant functions to model real-life data. For instance, Exercise 68 on page 306 shows you how a tangent function can be used to model and analyze the distance between a television camera and a parade unit.

A. Ramey/PhotoEdit

Example 1 Sketching the Graph of a Tangent Function

Sketch the graph of $y = \tan \frac{x}{2}$ by hand.

Solution

By solving the equations $x/2 = -\pi/2$ and $x/2 = \pi/2$, you can see that two consecutive asymptotes occur at $x = -\pi$ and $x = \pi$. Between these two asymptotes, plot a few points, including the x-intercept, as shown in the table. Three cycles of the graph are shown in Figure 4.56. Use a graphing utility to confirm this graph.

x	$-\pi$	$-\frac{\pi}{2}$	0	$\frac{\pi}{2}$	π
$\tan \frac{x}{2}$	Undef.	-1	0	1	Undef.

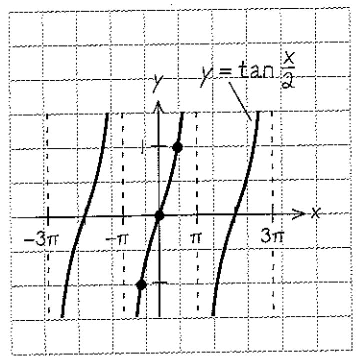


Figure 4.56

✓ *Checkpoint* Now try Exercise 7.

Example 2 Sketching the Graph of a Tangent Function

Sketch the graph of $y = -3 \tan 2x$.

Solution

By solving the equations $2x = -\pi/2$ and $2x = \pi/2$, you can see that two consecutive asymptotes occur at $x = -\pi/4$ and $x = \pi/4$. Between these two asymptotes, plot a few points, including the x-intercept, as shown in the table. Three complete cycles of the graph are shown in Figure 4.57.

x	$-\frac{\pi}{4}$	$-\frac{\pi}{8}$	0	$\frac{\pi}{8}$	$\frac{\pi}{4}$
$-3 \tan 2x$	Undef.	3	0	-3	Undef.

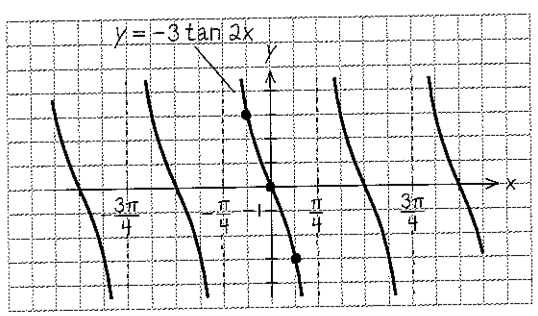


Figure 4.57

✓ *Checkpoint* Now try Exercise 9.

TECHNOLOGY TIP

Your graphing utility may connect parts of the graphs of tangent, cotangent, secant, and cosecant functions that are not supposed to be connected. So, in this text, these functions are graphed on a graphing utility using the *dot* mode. A blue curve is placed behind the graphing utility's display to indicate where the graph should appear. For instructions on how to use the *dot* mode, see Appendix A; for specific keystrokes, go to the text website at *college.hmco.com*.

TECHNOLOGY TIP Graphing utilities are helpful in verifying sketches of trigonometric functions. You can use a graphing utility set in *radian* and *dot* modes to graph the function $y = -3 \tan 2x$ from Example 2, as shown in Figure 4.58. You can use the *zero* or *root* feature or the *zoom* and *trace* features to approximate the key points of the graph.

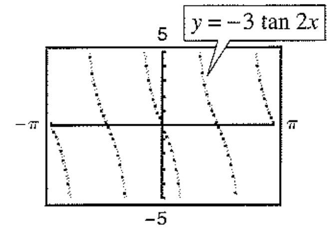


Figure 4.58

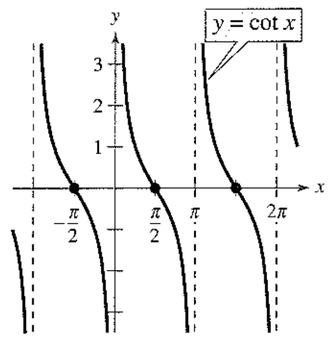
By comparing the graphs in Examples 1 and 2, you can see that the graph of $y = a \tan(bx - c)$ increases between consecutive vertical asymptotes when $a > 0$ and decreases between consecutive vertical asymptotes when $a < 0$. In other words, the graph for $a < 0$ is a reflection in the x-axis of the graph for $a > 0$.

Graph of the Cotangent Function

The graph of the cotangent function is similar to the graph of the tangent function. It also has a period of π. However, from the identity

$$y = \cot x = \frac{\cos x}{\sin x}$$

you can see that the cotangent function has vertical asymptotes when $\sin x$ is zero, which occurs at $x = n\pi$, where n is an integer. The graph of the cotangent function is shown in Figure 4.59.



Period: π
Domain: all $x \neq n\pi$
Range: $(-\infty, \infty)$
Vertical asymptotes: $x = n\pi$
Figure 4.59

Example 3 Sketching the Graph of a Cotangent Function

Sketch the graph of $y = 2 \cot \frac{x}{3}$ by hand.

Solution

To locate two consecutive vertical asymptotes of the graph, solve the equations $x/3 = 0$ and $x/3 = \pi$ to see that two consecutive asymptotes occur at $x = 0$ and $x = 3\pi$. Then, between these two asymptotes, plot a few points, including the x-intercept, as shown in the table. Three cycles of the graph are shown in Figure 4.60. Use a graphing utility to confirm this graph. [Enter the function as $y = 2/\tan(x/3)$.] Note that the period is 3π, the distance between consecutive asymptotes.

x	0	$\frac{3\pi}{4}$	$\frac{3\pi}{2}$	$\frac{9\pi}{4}$	3π
$2 \cot \frac{x}{3}$	Undef.	2	0	-2	Undef.

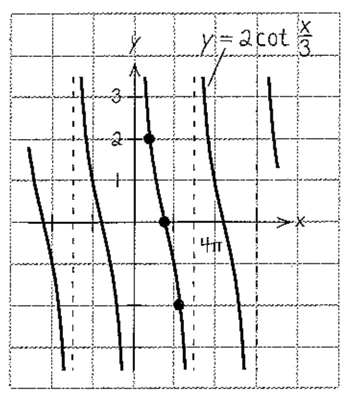


Figure 4.60

Checkpoint Now try Exercise 19.

Exploration

Use a graphing utility to graph the functions $y_1 = \cos x$ and $y_2 = \sec x = 1/\cos x$ in the same viewing window. How are the graphs related? What happens to the graph of the secant function as x approaches the zeros of the cosine function?

Graphs of the Reciprocal Functions

The graphs of the two remaining trigonometric functions can be obtained from the graphs of the sine and cosine functions using the reciprocal identities

$$\csc x = \frac{1}{\sin x} \quad \text{and} \quad \sec x = \frac{1}{\cos x}.$$

For instance, at a given value of x, the y-coordinate for $\sec x$ is the reciprocal of the y-coordinate for $\cos x$. Of course, when $\cos x = 0$, the reciprocal does not exist. Near such values of x, the behavior of the secant function is similar to that of the tangent function. In other words, the graphs of

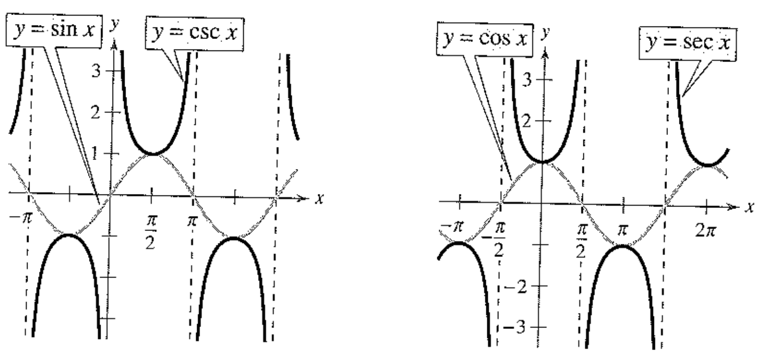
$$\tan x = \frac{\sin x}{\cos x} \quad \text{and} \quad \sec x = \frac{1}{\cos x}$$

have vertical asymptotes at $x = \pi/2 + n\pi$, where n is an integer (i.e., the values at which the cosine is zero). Similarly,

$$\cot x = \frac{\cos x}{\sin x} \quad \text{and} \quad \csc x = \frac{1}{\sin x}$$

have vertical asymptotes where $\sin x = 0$, that is, at $x = n\pi$.

To sketch the graph of a secant or cosecant function, you should first make a sketch of its reciprocal function. For instance, to sketch the graph of $y = \csc x$, first sketch the graph of $y = \sin x$. Then take the reciprocals of the y-coordinates to obtain points on the graph of $y = \csc x$. You can use this procedure to obtain the graphs shown in Figure 4.61.



Period: 2π

Domain: all $x \neq n\pi$

Range: $(-\infty, -1] \cup [1, \infty)$

Vertical asymptotes: $x = n\pi$

Symmetry: origin

Period: 2π

Domain: all $x \neq \dfrac{\pi}{2} + n\pi$

Range: $(-\infty, -1] \cup [1, \infty)$

Vertical asymptotes: $x = \dfrac{\pi}{2} + n\pi$

Symmetry: y-axis

Figure 4.61

In comparing the graphs of the secant and cosecant functions with those of the sine and cosine functions, note that the "hills" and "valleys" are interchanged. For example, a hill (or maximum point) on the sine curve corresponds to a valley (a local minimum) on the cosecant curve, and a valley (or minimum point) on the sine curve corresponds to a hill (a local maximum) on the cosecant curve, as shown in Figure 4.62. Additionally, x-intercepts of the sine and cosine functions become vertical asymptotes of the cosecant and secant functions, respectively (see Figure 4.62).

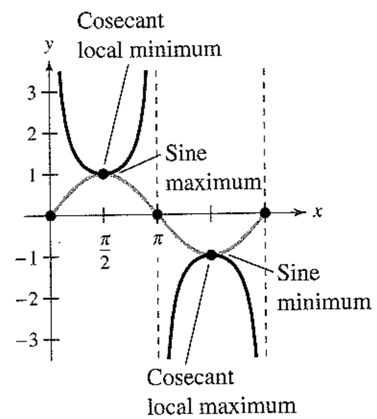


Figure 4.62

Example 4 Comparing Trigonometric Graphs

Use a graphing utility to compare the graphs of

$$y = 2\sin\left(x + \frac{\pi}{4}\right) \quad \text{and} \quad y = 2\csc\left(x + \frac{\pi}{4}\right).$$

Solution

The two graphs are shown in Figure 4.63. Note how the hills and valleys of the graphs are related. For the function $y = 2\sin[x + (\pi/4)]$, the amplitude is 2 and the period is 2π. By solving the equations

$$x + \frac{\pi}{4} = 0 \quad \text{and} \quad x + \frac{\pi}{4} = 2\pi$$

you can see that one cycle of the sine function corresponds to the interval from $x = -\pi/4$ to $x = 7\pi/4$. The graph of this sine function is represented by the thick curve in Figure 4.63. Because the sine function is zero at the endpoints of this interval, the corresponding cosecant function

$$y = 2\csc\left(x + \frac{\pi}{4}\right) = 2\left(\frac{1}{\sin[x + (\pi/4)]}\right)$$

has vertical asymptotes at $x = -\pi/4, 3\pi/4, 7\pi/4$, and so on.

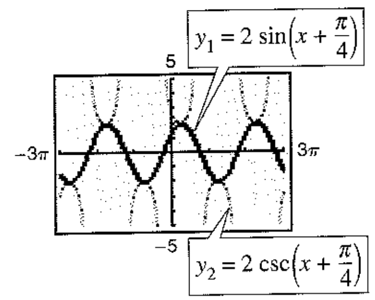


Figure 4.63

✓ *Checkpoint* Now try Exercise 31.

Example 5 Comparing Trigonometric Graphs

Use a graphing utility to compare the graphs of $y = \cos 2x$ and $y = \sec 2x$.

Solution

Begin by graphing $y_1 = \cos 2x$ and $y_2 = \sec 2x = 1/\cos 2x$ in the same viewing window, as shown in Figure 4.64. Note that the x-intercepts of $y = \cos 2x$

$$\left(-\frac{\pi}{4}, 0\right), \quad \left(\frac{\pi}{4}, 0\right), \quad \left(\frac{3\pi}{4}, 0\right), \ldots$$

correspond to the vertical asymptotes

$$x = -\frac{\pi}{4}, \quad x = \frac{\pi}{4}, \quad x = \frac{3\pi}{4}, \ldots$$

of the graph of $y = \sec 2x$. Moreover, notice that the period of $y = \cos 2x$ and $y = \sec 2x$ is π.

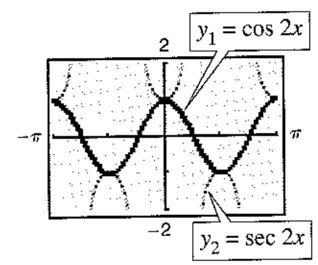


Figure 4.64

✓ *Checkpoint* Now try Exercise 33.

Damped Trigonometric Graphs

A *product* of two functions can be graphed using properties of the individual functions. For instance, consider the function

$f(x) = x \sin x$

as the product of the functions $y = x$ and $y = \sin x$. Using properties of absolute value and the fact that $|\sin x| \le 1$, you have $0 \le |x|\,|\sin x| \le |x|$. Consequently,

$-|x| \le x \sin x \le |x|$

which means that the graph of $f(x) = x \sin x$ lies between the lines $y = -x$ and $y = x$. Furthermore, because

$f(x) = x \sin x = \pm x \quad \text{at} \quad x = \frac{\pi}{2} + n\pi$

and

$f(x) = x \sin x = 0 \quad \text{at} \quad x = n\pi$

the graph of f touches the line $y = -x$ or the line $y = x$ at $x = \pi/2 + n\pi$ and has x-intercepts at $x = n\pi$. A sketch of f is shown in Figure 4.65. In the function $f(x) = x \sin x$, the factor x is called the **damping factor.**

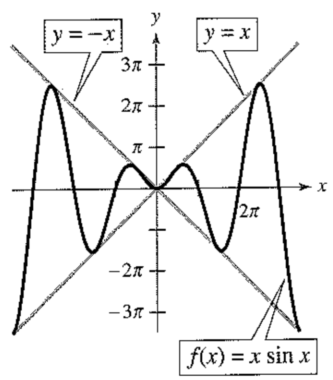


Figure 4.65

Example 6 Analyzing a Damped Sine Curve

Analyze the graph of

$f(x) = e^{-x} \sin 3x.$

Solution

Consider $f(x)$ as the product of the two functions

$y = e^{-x} \quad \text{and} \quad y = \sin 3x$

each of which has the set of real numbers as its domain. For any real number x, you know that $e^{-x} \ge 0$ and $|\sin 3x| \le 1$. So, $|e^{-x}|\,|\sin 3x| \le e^{-x}$, which means that

$-e^{-x} \le e^{-x} \sin 3x \le e^{-x}.$

Furthermore, because

$f(x) = e^{-x} \sin 3x = \pm e^{-x} \quad \text{at} \quad x = \frac{\pi}{6} + \frac{n\pi}{3}$

and

$f(x) = e^{-x} \sin 3x = 0 \quad \text{at} \quad x = \frac{n\pi}{3}$

the graph of f touches the curves $y = -e^{-x}$ and $y = e^{-x}$ at $x = \pi/6 + n\pi/3$ and has intercepts at $x = n\pi/3$. The graph is shown in Figure 4.66.

✓ *Checkpoint* Now try Exercise 57.

STUDY TIP

Do you see why the graph of $f(x) = x \sin x$ touches the lines $y = \pm x$ at $x = \pi/2 + n\pi$ and why the graph has x-intercepts at $x = n\pi$? Recall that the sine function is equal to ± 1 at $\pi/2, 3\pi/2, 5\pi/2, \ldots$ (odd multiples of $\pi/2$) and is equal to 0 at $\pi, 2\pi, 3\pi, \ldots$ (multiples of π).

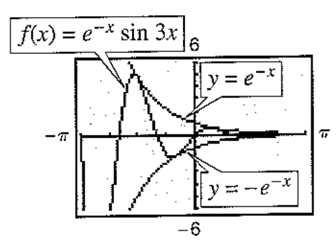
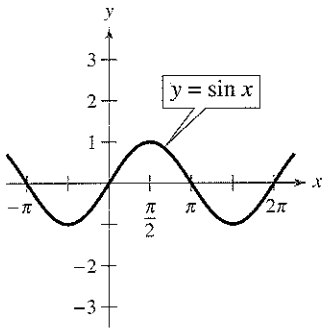
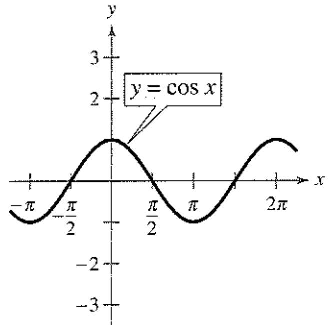


Figure 4.66

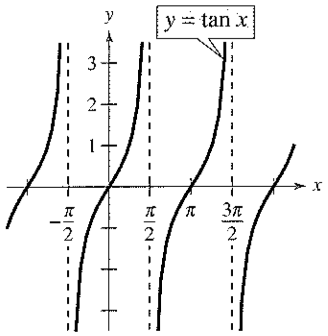
Figure 4.67 summarizes the six basic trigonometric functions.



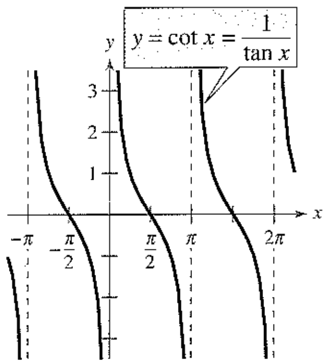
Domain: all reals
Range: $[-1, 1]$
Period: 2π



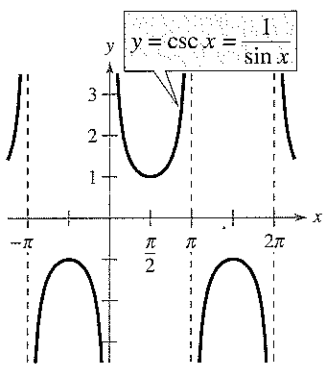
Domain: all reals
Range: $[-1, 1]$
Period: 2π



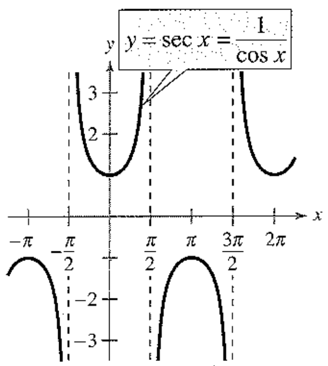
Domain: all $x \neq \frac{\pi}{2} + n\pi$
Range: $(-\infty, \infty)$
Period: π



Domain: all $x \neq n\pi$
Range: $(-\infty, \infty)$
Period: π



Domain: all $x \neq n\pi$
Range: $(-\infty, -1] \cup [1, \infty)$
Period: 2π



Domain: all $x \neq \frac{\pi}{2} + n\pi$
Range: $(-\infty, -1] \cup [1, \infty)$
Period: 2π

Figure 4.67

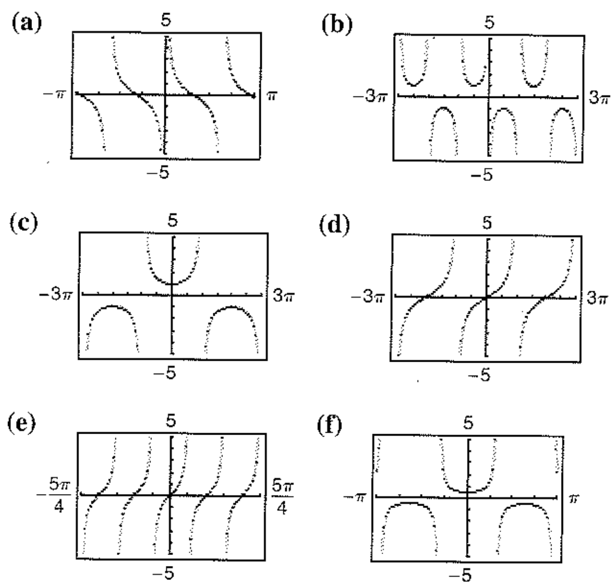
4.6 Exercises

Vocabulary Check

Fill in the blanks.

1. The graphs of the tangent, cotangent, secant, and cosecant functions have ________ asymptotes.
2. To sketch the graph of a secant or cosecant function, first make a sketch of its ________ function.
3. For the function $f(x) = g(x) \sin x$, $g(x)$ is called the ________ factor of the function.

In Exercises 1–6, match the function with its graph. State the period of the function. [The graphs are labeled (a), (b), (c), (d), (e), and (f).]



1. $y = \sec \dfrac{x}{2}$
2. $y = \tan \dfrac{x}{2}$
3. $y = \tan 2x$
4. $y = \dfrac{1}{2} \sec \dfrac{\pi x}{2}$
5. $y = \cot \dfrac{\pi x}{2}$
6. $y = -\csc x$

In Exercises 7–30, sketch the graph of the function. (Include two full periods.) Use a graphing utility to verify your result.

7. $y = \frac{1}{2} \tan x$
8. $y = \frac{1}{4} \tan x$
9. $y = -2 \tan 2x$
10. $y = -3 \tan 4x$
11. $y = -\frac{1}{2} \sec x$
12. $y = \frac{1}{4} \sec x$
13. $y = -\sec \pi x$
14. $y = 2 \sec \pi x$
15. $y = \sec \pi x - 3$
16. $y = -2 \sec 4x + 2$
17. $y = 3 \csc \dfrac{x}{2}$
18. $y = -\csc \dfrac{x}{3}$
19. $y = \dfrac{1}{2} \cot \dfrac{x}{2}$
20. $y = 3 \cot \pi x$
21. $y = 2 \tan \dfrac{\pi x}{4}$
22. $y = -\dfrac{1}{2} \tan \pi x$
23. $y = \frac{1}{2} \sec 2x$
24. $y = \sec(x + \pi)$
25. $y = \csc(\pi - x)$
26. $y = \csc(2x - \pi)$
27. $y = 2 \cot\left(x - \dfrac{\pi}{2}\right)$
28. $y = \dfrac{1}{4} \cot(x + \pi)$
29. $y = \tan\left(x - \dfrac{\pi}{4}\right)$
30. $y = \dfrac{1}{2} \tan\left(\dfrac{\pi x}{4} + \dfrac{\pi}{4}\right)$

In Exercises 31–36, use a graphing utility to graph the function. (Include two full periods.) Compare the graph of the function with the graph of the corresponding reciprocal function. Describe your viewing window.

31. $y = 2 \csc 3x$
32. $y = -\csc(4x - \pi)$
33. $y = -2 \sec 4x$
34. $y = \frac{1}{4} \sec \pi x$
35. $y = \dfrac{1}{3} \sec\left(\dfrac{\pi x}{2} + \dfrac{\pi}{2}\right)$
36. $y = \dfrac{1}{2} \csc(2x - \pi)$

In Exercises 37–40, use a graph to solve the equation on the interval $[-2\pi, 2\pi]$.

37. $\tan x = 1$
38. $\cot x = -\sqrt{3}$
39. $\sec x = -2$
40. $\csc x = \sqrt{2}$

In Exercises 41–44, use the graph of the function to determine whether the function is even, odd, or neither.

41. $f(x) = \sec x$
42. $f(x) = \tan x$
43. $f(x) = \csc 2x$
44. $f(x) = \cot 2x$

In Exercises 45–48, use a graphing utility to graph the two equations in the same viewing window. Use the graphs to determine whether the expressions are equivalent. Verify the results algebraically.

45. $y_1 = \sin x \csc x, \quad y_2 = 1$

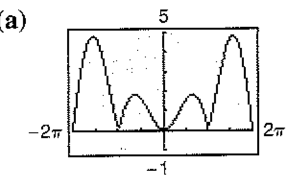
46. $y_1 = \sin x \sec x, \quad y_2 = \tan x$

47. $y_1 = \dfrac{\cos x}{\sin x}, \quad y_2 = \cot x$

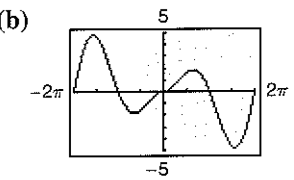
48. $y_1 = \sec^2 x - 1, \quad y_2 = \tan^2 x$

In Exercises 49–52, match the function with its graph. Describe the behavior of the function as x approaches zero. [The graphs are labeled (a), (b), (c), and (d).]

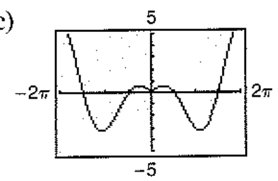
(a)



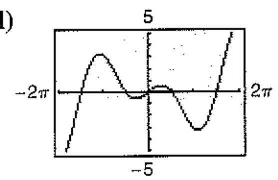
(b)



(c)



(d)



49. $f(x) = x \cos x$

50. $f(x) = |x \sin x|$

51. $g(x) = |x| \sin x$

52. $g(x) = |x| \cos x$

Conjecture **In Exercises 53–56, use a graphing utility to graph the functions f and g. Use the graphs to make a conjecture about the relationship between the functions.**

53. $f(x) = \sin x + \cos\left(x + \dfrac{\pi}{2}\right), \quad g(x) = 0$

54. $f(x) = \sin x - \cos\left(x + \dfrac{\pi}{2}\right), \quad g(x) = 2 \sin x$

55. $f(x) = \sin^2 x, \quad g(x) = \frac{1}{2}(1 - \cos 2x)$

56. $f(x) = \cos^2 \dfrac{\pi x}{2}, \quad g(x) = \dfrac{1}{2}(1 + \cos \pi x)$

In Exercises 57–60, use a graphing utility to graph the function and the damping factor of the function in the same viewing window. Describe the behavior of the function as x increases without bound.

57. $f(x) = e^{-x} \cos x$

58. $f(x) = 2^{-x/4} \cos \pi x$

59. $g(x) = e^{-x^2/2} \sin x$

60. $h(x) = 2^{-x^2/4} \sin x$

Exploration **In Exercises 61–66, use a graphing utility to graph the function. Describe the behavior of the function as x approaches zero.**

61. $f(x) = \dfrac{6}{x} + \cos x$

62. $f(x) = \sin x - \dfrac{4}{x}$

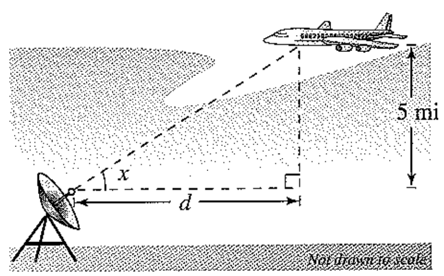
63. $f(x) = \dfrac{\sin x}{x}$

64. $f(x) = \dfrac{1 - \cos x}{x}$

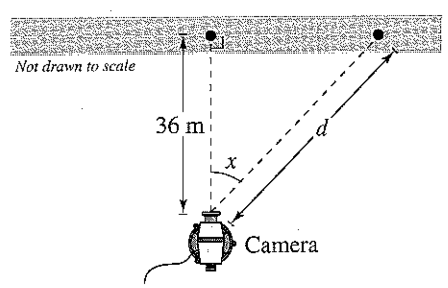
65. $f(x) = \dfrac{\tan x}{x}$

66. $f(x) = \dfrac{x}{\cot x}$

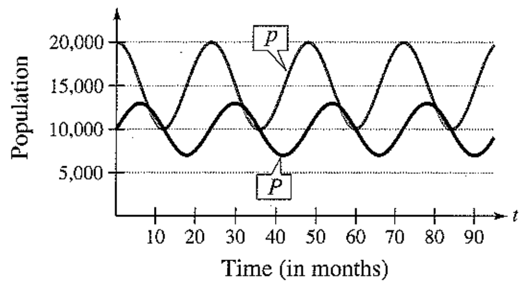
67. ***Distance*** A plane flying at an altitude of 5 miles over level ground will pass directly over a radar antenna (see figure). Let d be the ground distance from the antenna to the point directly under the plane and let x be the angle of elevation to the plane from the antenna. (d is positive as the plane approaches the antenna.) Write d as a function of x and graph the function over the interval $0 < x < \pi$.



68. ***Television Coverage*** A television camera is on a reviewing platform 36 meters from the street on which a parade will be passing from left to right (see figure). Write the distance d from the camera to a particular unit in the parade as a function of the angle x, and graph the function over the interval $-\pi/2 < x < \pi/2$. (Consider x as negative when a unit in the parade approaches from the left.)



69. ***Predator-Prey Model*** The population of coyotes (a predator) at time t (in months) in a region is estimated to be $P = 10{,}000 + 3000 \sin(\pi t/12)$ and the population of rabbits (its prey) is estimated to be $p = 15{,}000 + 5000 \cos(\pi t/12)$. Use the graph of the models to explain the oscillations in the size of each population.



70. ***Meteorology*** The normal monthly high temperatures in degrees Fahrenheit for Erie, Pennsylvania, are approximated by

$$H(t) = 54.33 - 20.38 \cos \frac{\pi t}{6} - 15.69 \sin \frac{\pi t}{6}$$

and the normal monthly low temperatures are approximated by

$$L(t) = 39.36 - 15.70 \cos \frac{\pi t}{6} - 14.16 \sin \frac{\pi t}{6}$$

where t is the time (in months), with $t = 1$ corresponding to January. (Source: National Oceanic and Atmospheric Association)

(a) Use a graphing utility to graph each function. What is the period of each function?

(b) During what part of the year is the difference between the normal high and normal low temperatures greatest? When is it smallest?

(c) The sun is the farthest north in the sky around June 21, but the graph shows the warmest temperatures at a later date. Approximate the lag time of the temperatures relative to the position of the sun.

71. ***Harmonic Motion*** An object weighing W pounds is suspended from a ceiling by a steel spring (see figure). The weight is pulled downward (positive direction) from its equilibrium position and released. The resulting motion of the weight is described by the function $y = \frac{1}{2}e^{-t/4} \cos 4t$, where y is the distance in feet and t is the time in seconds $(t > 0)$.

(a) Use a graphing utility to graph the function.

(b) Describe the behavior of the displacement function for increasing values of time t.

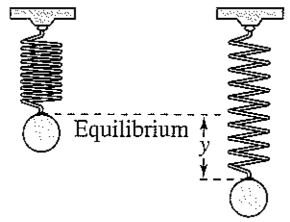
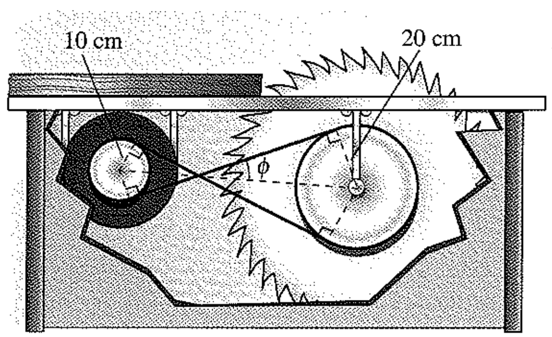


Figure for 71

72. ***Numerical and Graphical Reasoning*** A crossed belt connects a 10-centimeter pulley on an electric motor with a 20-centimeter pulley on a saw arbor (see figure). The electric motor runs at 1700 revolutions per minute.



(a) Determine the number of revolutions per minute of the saw.

(b) How does crossing the belt affect the saw in relation to the motor?

(c) Let L be the total length of the belt. Write L as a function of ϕ, where ϕ is measured in radians. What is the domain of the function? (*Hint:* Add the lengths of the straight sections of the belt and the length of belt around each pulley.)

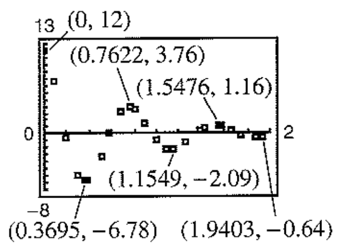
(d) Use a graphing utility to complete the table.

ϕ	0.3	0.6	0.9	1.2	1.5
L					

(e) As ϕ increases, do the lengths of the straight sections of the belt change faster or slower than the lengths of the belts around each pulley?

(f) Use a graphing utility to graph the function over the appropriate domain.

73. ***Data Analysis*** The motion of an oscillating weight suspended by a spring was measured by a motion detector. The data was collected, and the approximate maximum (positive and negative) displacements from equilibrium are shown in the graph. The displacement y is measured in centimeters and the time t is measured in seconds.



(a) Is y a function of t? Explain.

(b) Approximate the frequency of the oscillations.

(c) Fit a model of the form $y = ab^t \cos ct$ to the data. Use the result of part (b) to approximate c. Use the *regression* feature of a graphing utility to fit an exponential model to the positive maximum displacements of the weight.

(d) Rewrite the model in the form $y = ae^{kt} \cos ct$.

(e) Use a graphing utility to graph the model. Compare the result with the data in the graph above.

74. ***Writing*** Write a short paragraph describing the specified change in the physical system of Exercise 73.

(a) A spring of less stiffness is used, and so the length of time for each oscillation is greater.

(b) The effect of friction is decreased.

Synthesis

True or False? **In Exercises 75 and 76, determine whether the statement is true or false. Justify your answer.**

75. The graph of $y = -\frac{1}{8}\tan\left(\frac{x}{2} + \pi\right)$ has an asymptote at $x = -3\pi$.

76. For the graph of $y = 2^x \sin x$, as x approaches $-\infty$, y approaches 0.

77. ***Writing*** Describe the behavior of $f(x) = \tan x$ as x approaches $\pi/2$ from the left and from the right.

78. ***Writing*** Describe the behavior of $f(x) = \csc x$ as x approaches π from the left and from the right.

79. ***Graphical Reasoning*** Consider the functions $f(x) = 2 \sin x$ and $g(x) = \frac{1}{2}\csc x$ on the interval $(0, \pi)$.

(a) Use a graphing utility to graph f and g in the same viewing window.

(b) Approximate the interval in which $f > g$.

(c) Describe the behavior of each of the functions as x approaches π. How is the behavior of g related to the behavior of f as x approaches π?

80. ***Pattern Recognition***

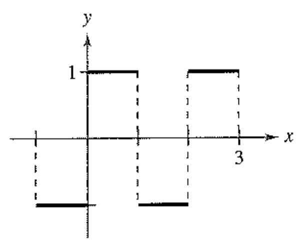
(a) Use a graphing utility to graph each function.

$$y_1 = \frac{4}{\pi}\left(\sin \pi x + \frac{1}{3}\sin 3\pi x\right)$$

$$y_2 = \frac{4}{\pi}\left(\sin \pi x + \frac{1}{3}\sin 3\pi x + \frac{1}{5}\sin 5\pi x\right)$$

(b) Identify the pattern in part (a) and find a function y_3 that continues the pattern one more term. Use a graphing utility to graph y_3.

(c) The graphs in parts (a) and (b) approximate the periodic function in the figure. Find a function y_4 that is a better approximation.



Review

In Exercises 81–84, identify the rule of algebra illustrated by the statement.

81. $5(a - 9) = 5a - 45$ **82.** $7\left(\frac{1}{7}\right) = 1$

83. $(3 + x) + 0 = 3 + x$

84. $(a + b) + 10 = a + (b + 10)$

In Exercises 85–88, determine whether the function is one-to-one. If it is, find its inverse function.

85. $f(x) = -10$ **86.** $f(x) = (x - 7)^2 + 3$

87. $f(x) = \sqrt{3x - 14}$ **88.** $f(x) = \sqrt[3]{x - 5}$

4.7 Inverse Trigonometric Functions

Inverse Sine Function

Recall from Section 1.6 that for a function to have an inverse function, it must be one-to-one—that is, it must pass the Horizontal Line Test. From Figure 4.68 it is obvious that $y = \sin x$ does not pass the test because different values of x yield the same y-value.

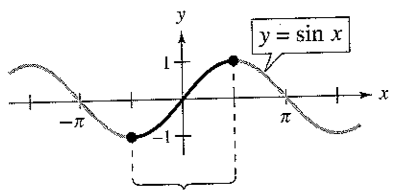


Figure 4.68

However, if you restrict the domain to the interval $-\pi/2 \le x \le \pi/2$ (corresponding to the black portion of the graph in Figure 4.68), the following properties hold.

1. On the interval $[-\pi/2, \pi/2]$, the function $y = \sin x$ is increasing.
2. On the interval $[-\pi/2, \pi/2]$, $y = \sin x$ takes on its full range of values, $-1 \le \sin x \le 1$.
3. On the interval $[-\pi/2, \pi/2]$, $y = \sin x$ is one-to-one.

So, on the restricted domain $-\pi/2 \le x \le \pi/2$, $y = \sin x$ has a unique inverse function called the **inverse sine function.** It is denoted by

$$y = \arcsin x \quad \text{or} \quad y = \sin^{-1} x.$$

The notation $\sin^{-1} x$ is consistent with the inverse function notation $f^{-1}(x)$. The $\arcsin x$ notation (read as "the arcsine of x") comes from the association of a central angle with its intercepted *arc length* on a unit circle. So, $\arcsin x$ means the angle (or arc) whose sine is x. Both notations, $\arcsin x$ and $\sin^{-1} x$, are commonly used in mathematics, so remember that $\sin^{-1} x$ denotes the *inverse* sine function rather than $1/\sin x$. The values of $\arcsin x$ lie in the interval $-\pi/2 \le \arcsin x \le \pi/2$. The graph of $y = \arcsin x$ is shown in Example 2.

Definition of Inverse Sine Function

The **inverse sine function** is defined by

$$y = \arcsin x \quad \text{if and only if} \quad \sin y = x$$

where $-1 \le x \le 1$ and $-\pi/2 \le y \le \pi/2$. The domain of $y = \arcsin x$ is $[-1, 1]$ and the range is $[-\pi/2, \pi/2]$.

When evaluating the inverse sine function, it helps to remember the phrase "the arcsine of x is the angle (or number) whose sine is x."

What you should learn

- Evaluate inverse sine functions.
- Evaluate other inverse trigonometric functions.
- Evaluate compositions of trigonometric functions.

Why you should learn it

Inverse trigonometric functions can be useful in exploring how two aspects of a real-life problem relate to each other. Exercise 71 on page 318 investigates the relationship between the length of rope from a winch to a boat and the angle of elevation between them.

Arlene Collins

Example 1 Evaluating the Inverse Sine Function

If possible, find the exact value.

a. $\arcsin\left(-\frac{1}{2}\right)$ **b.** $\sin^{-1}\frac{\sqrt{3}}{2}$ **c.** $\sin^{-1} 2$

Solution

a. Because $\sin\left(-\frac{\pi}{6}\right) = -\frac{1}{2}$, and $-\frac{\pi}{6}$ lies in $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$, it follows that

$$\arcsin\left(-\frac{1}{2}\right) = -\frac{\pi}{6}. \qquad \text{Angle whose sine is } -\tfrac{1}{2}$$

b. Because $\sin\left(\frac{\pi}{3}\right) = \frac{\sqrt{3}}{2}$, and $\frac{\pi}{3}$ lies in $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$, it follows that

$$\sin^{-1}\frac{\sqrt{3}}{2} = \frac{\pi}{3}. \qquad \text{Angle whose sine is } \sqrt{3}/2$$

c. It is not possible to evaluate $y = \sin^{-1} x$ at $x = 2$ because there is no angle whose sine is 2. Remember that the domain of the inverse sine function is $[-1, 1]$.

✓ *Checkpoint* Now try Exercise 1.

STUDY TIP

As with the trigonometric functions, much of the work with the inverse trigonometric functions can be done by *exact* calculations rather than by calculator approximations. Exact calculations help to increase your understanding of the inverse functions by relating them to the triangle definitions of the trigonometric functions.

Example 2 Graphing the Arcsine Function

Sketch a graph of $y = \arcsin x$ by hand.

Solution

By definition, the equations

$$y = \arcsin x \qquad \text{and} \qquad \sin y = x$$

are equivalent for $-\pi/2 \le y \le \pi/2$. So, their graphs are the same. For the interval $[-\pi/2, \pi/2]$, you can assign values to y in the second equation to make a table of values.

y	$-\frac{\pi}{2}$	$-\frac{\pi}{4}$	$-\frac{\pi}{6}$	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{2}$
$x = \sin y$	-1	$-\frac{\sqrt{2}}{2}$	$-\frac{1}{2}$	0	$\frac{1}{2}$	$\frac{\sqrt{2}}{2}$	1

Then plot the points and connect them with a smooth curve. The resulting graph of $y = \arcsin x$ is shown in Figure 4.69. Note that it is the reflection (in the line $y = x$) of the black portion of the graph in Figure 4.68. Use a graphing utility to confirm this graph. Be sure you see that Figure 4.69 shows the *entire* graph of the inverse sine function. Remember that the domain of $y = \arcsin x$ is the closed interval $[-1, 1]$ and the range is the closed interval $[-\pi/2, \pi/2]$.

✓ *Checkpoint* Now try Exercise 8.

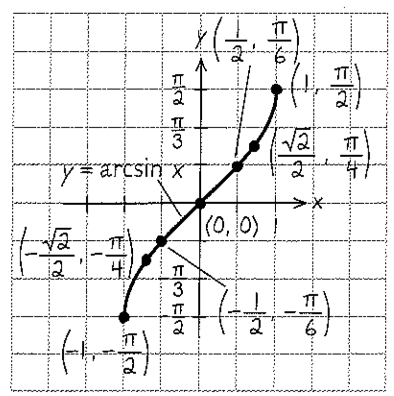


Figure 4.69

Other Inverse Trigonometric Functions

The cosine function is decreasing and one-to-one on the interval $0 \le x \le \pi$, as shown in Figure 4.70.

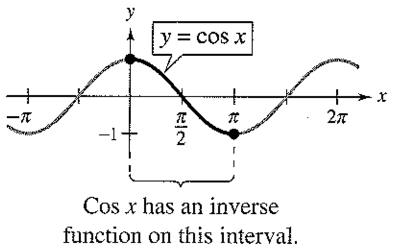


Figure 4.70

Consequently, on this interval the cosine function has an inverse function—the **inverse cosine function**—denoted by

$$y = \arccos x \quad \text{or} \quad y = \cos^{-1} x.$$

Because $y = \arccos x$ and $x = \cos y$ are equivalent for $0 \le y \le \pi$, their graphs are the same, and can be confirmed by the following table of values.

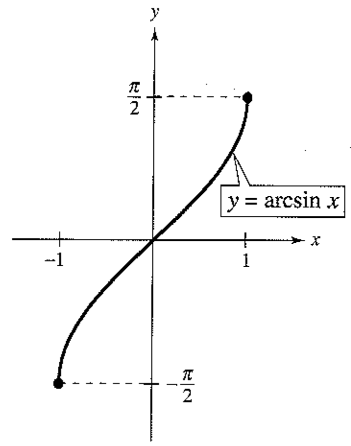
y	0	$\frac{\pi}{6}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{5\pi}{6}$	π
$x = \cos y$	1	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$	0	$-\frac{1}{2}$	$-\frac{\sqrt{3}}{2}$	-1

Similarly, you can define an **inverse tangent function** by restricting the domain of $y = \tan x$ to the interval $(-\pi/2, \pi/2)$. The following list summarizes the definitions of the three most common inverse trigonometric functions. The remaining three are defined in Exercises 79–81.

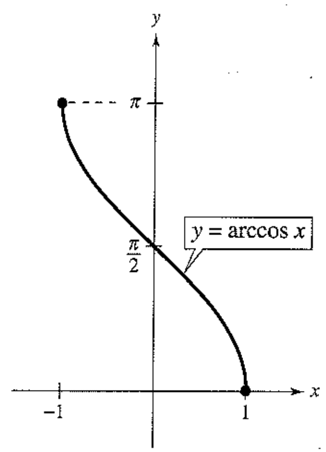
Definition of the Inverse Trigonometric Functions

Function	*Domain*	*Range*
$y = \arcsin x$ if and only if $\sin y = x$	$-1 \le x \le 1$	$-\frac{\pi}{2} \le y \le \frac{\pi}{2}$
$y = \arccos x$ if and only if $\cos y = x$	$-1 \le x \le 1$	$0 \le y \le \pi$
$y = \arctan x$ if and only if $\tan y = x$	$-\infty < x < \infty$	$-\frac{\pi}{2} < y < \frac{\pi}{2}$

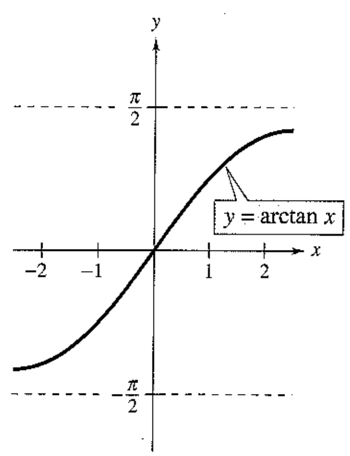
The graphs of these three inverse trigonometric functions are shown in Figure 4.71.



Domain: $[-1, 1]$; Range: $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$



Domain: $[-1, 1]$; Range: $[0, \pi]$



Domain: $(-\infty, \infty)$; Range: $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

Figure 4.71

Example 3 Evaluating Inverse Trigonometric Functions

Find the exact value.

a. $\arccos \frac{\sqrt{2}}{2}$ **b.** $\cos^{-1}(-1)$ **c.** $\arctan 0$ **d.** $\tan^{-1}(-1)$

Solution

a. Because $\cos(\pi/4) = \sqrt{2}/2$, and $\pi/4$ lies in $[0, \pi]$, it follows that

$$\arccos \frac{\sqrt{2}}{2} = \frac{\pi}{4}. \qquad \text{Angle whose cosine is } \frac{\sqrt{2}}{2}$$

b. Because $\cos \pi = -1$, and π lies in $[0, \pi]$, it follows that

$$\cos^{-1}(-1) = \pi. \qquad \text{Angle whose cosine is } -1$$

c. Because $\tan 0 = 0$, and 0 lies in $(-\pi/2, \pi/2)$, it follows that

$$\arctan 0 = 0. \qquad \text{Angle whose tangent is } 0$$

d. Because $\tan(-\pi/4) = -1$ and $-\pi/4$ lies in $(-\pi/2, \pi/2)$, it follows that

$$\tan^{-1}(-1) = -\frac{\pi}{4}. \qquad \text{Angle whose tangent is } -1$$

Checkpoint Now try Exercise 3.

Example 4 Calculators and Inverse Trigonometric Functions

Use a calculator to approximate the value (if possible).

a. $\arctan(-8.45)$ **b.** $\sin^{-1} 0.2447$ **c.** $\arccos 2$

Solution

Function	*Mode*	*Graphing Calculator Keystrokes*
a. $\arctan(-8.45)$	Radian	[TAN⁻¹] [(-)] 8.45 [)] [ENTER]

From the display, it follows that $\arctan(-8.45) \approx -1.4530010$.

b. $\sin^{-1} 0.2447$	Radian	[SIN⁻¹] 0.2447 [ENTER]

From the display, it follows that $\sin^{-1} 0.2447 \approx 0.2472103$.

c. $\arccos 2$	Radian	[COS⁻¹] 2 [ENTER]

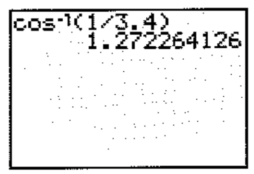
In *real number* mode, the calculator should display an *error message* because the domain of the inverse cosine function is $[-1, 1]$.

Checkpoint Now try Exercise 13.

TECHNOLOGY TIP In Example 4, if you had set the calculator to *degree* mode, the display would have been in degrees rather than in radians. This convention is peculiar to calculators. By definition, the values of inverse trigonometric functions are always in *radians*.

TECHNOLOGY TIP

You can use the [SIN⁻¹], [COS⁻¹], and [TAN⁻¹] keys on your calculator to approximate values of inverse trigonometric functions. To evaluate the inverse cosecant function, the inverse secant function, or the inverse cotangent function, you can use the inverse sine, inverse cosine, and inverse tangent functions, respectively. For instance, to evaluate $\sec^{-1} 3.4$, enter the expression as shown below.



Compositions of Functions

Recall from Section 1.6 that for all x in the domains of f and f^{-1}, inverse functions have the properties

$$f(f^{-1}(x)) = x \quad \text{and} \quad f^{-1}(f(x)) = x.$$

Inverse Properties

If $-1 \le x \le 1$ and $-\pi/2 \le y \le \pi/2$, then

$$\sin(\arcsin x) = x \quad \text{and} \quad \arcsin(\sin y) = y.$$

If $-1 \le x \le 1$ and $0 \le y \le \pi$, then

$$\cos(\arccos x) = x \quad \text{and} \quad \arccos(\cos y) = y.$$

If x is a real number and $-\pi/2 < y < \pi/2$, then

$$\tan(\arctan x) = x \quad \text{and} \quad \arctan(\tan y) = y.$$

Keep in mind that these inverse properties do not apply for arbitrary values of x and y. For instance,

$$\arcsin\left(\sin \frac{3\pi}{2}\right) = \arcsin(-1) = -\frac{\pi}{2} \neq \frac{3\pi}{2}.$$

In other words, the property $\arcsin(\sin y) = y$ is not valid for values of y outside the interval $[-\pi/2, \pi/2]$.

Exploration

Use a graphing utility to graph $y = \arcsin(\sin x)$. What are the domain and range of this function? Explain why $\arcsin(\sin 4)$ does not equal 4.

Now graph $y = \sin(\arcsin x)$ and determine the domain and range. Explain why $\sin(\arcsin 4)$ is not defined.

Example 5 Using Inverse Properties

If possible, find the exact value.

a. $\tan[\arctan(-5)]$ **b.** $\arcsin\left(\sin \frac{5\pi}{3}\right)$ **c.** $\cos(\cos^{-1} \pi)$

Solution

a. Because -5 lies in the domain of the arctan function, the inverse property applies, and you have $\tan[\arctan(-5)] = -5$.

b. In this case, $5\pi/3$ does not lie within the range of the arcsine function, $-\pi/2 \le y \le \pi/2$. However, $5\pi/3$ is coterminal with

$$\frac{5\pi}{3} - 2\pi = -\frac{\pi}{3}$$

which does lie in the range of the arcsine function, and you have

$$\arcsin\left(\sin \frac{5\pi}{3}\right) = \arcsin\left[\sin\left(-\frac{\pi}{3}\right)\right] = -\frac{\pi}{3}.$$

c. The expression $\cos(\cos^{-1} \pi)$ is not defined because $\cos^{-1} \pi$ is not defined. Remember that the domain of the inverse cosine function is $[-1, 1]$.

✓ ***Checkpoint*** Now try Exercise 33.

Example 6 shows how to use right triangles to find exact values of compositions of inverse functions.

Example 6 Evaluating Compositions of Functions

Find the exact value.

a. $\tan\left(\arccos \frac{2}{3}\right)$ **b.** $\cos\left[\arcsin\left(-\frac{3}{5}\right)\right]$

Algebraic Solution

a. If you let $u = \arccos \frac{2}{3}$, then $\cos u = \frac{2}{3}$. Because $\cos u$ is positive, u is a first-quadrant angle. You can sketch and label angle u as shown in Figure 4.72.

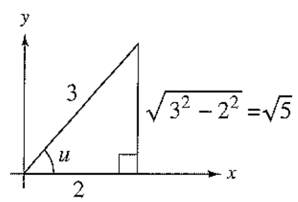


Figure 4.72

Consequently,

$$\tan\left(\arccos \frac{2}{3}\right) = \tan u = \frac{\text{opp}}{\text{adj}} = \frac{\sqrt{5}}{2}.$$

b. If you let $u = \arcsin\left(-\frac{3}{5}\right)$, then $\sin u = -\frac{3}{5}$. Because $\sin u$ is negative, u is a fourth-quadrant angle. You can sketch and label angle u as shown in Figure 4.73.

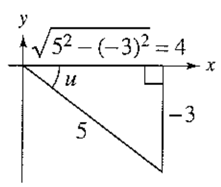


Figure 4.73

Consequently,

$$\cos\left[\arcsin\left(-\frac{3}{5}\right)\right] = \cos u = \frac{\text{adj}}{\text{hyp}} = \frac{4}{5}.$$

✓ ***Checkpoint*** Now try Exercise 41.

Graphical Solution

a. Use a graphing utility set in *radian* mode to graph $y = \tan(\arccos x)$, as shown in Figure 4.74. Use the *value* feature or the *zoom* and *trace* features of the graphing utility to find that the value of the composition of functions when $x = \frac{2}{3} \approx 0.66667$ is

$$y = 1.118 \approx \frac{\sqrt{5}}{2}.$$

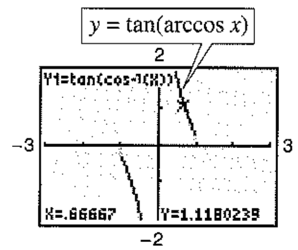


Figure 4.74

b. Use a graphing utility set in *radian* mode to graph $y = \cos(\arcsin x)$, as shown in Figure 4.75. Use the *value* feature or the *zoom* and *trace* features of the graphing utility to find that the value of the composition of functions when $x = -\frac{3}{5} = -0.6$ is

$$y = 0.8 = \frac{4}{5}.$$

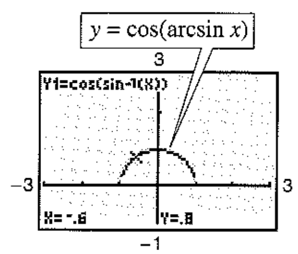


Figure 4.75

Library of Functions: Inverse Trigonometric Functions

The inverse trigonometric functions are obtained from the trigonometric functions in much the same way that the logarithmic function was developed from the exponential function. However, unlike the exponential function, the trigonometric functions are not one-to-one, and so it is necessary to restrict their domains to intervals on which they pass the Horizontal Line Test. Consequently, the inverse trigonometric functions have restricted domains and ranges, and they are not periodic.

One prominent role played by inverse trigonometric functions is in solving trigonometric equations in which the argument (angle) of the trigonometric function is the unknown quantity in the equation. You will learn how to solve such equations in the next chapter.

Inverse trigonometric functions play a unique role in calculus. There are two basic operations of calculus. One operation (called *differentiation*) transforms an inverse trigonometric function (a transcendental function) into an algebraic function. The other operation (called *integration*) produces the opposite transformation—from algebraic to transcendental.

Example 7 Some Problems from Calculus

algebra of calculus

Write each of the following as an algebraic expression in x.

a. $\sin(\arccos 3x), \quad 0 \le x \le \frac{1}{3}$ **b.** $\cot(\arccos 3x), \quad 0 \le x < \frac{1}{3}$

Solution

If you let $u = \arccos 3x$, then $\cos u = 3x$, where $-1 \le 3x \le 1$. Because

$$\cos u = \frac{\text{adj}}{\text{hyp}} = \frac{3x}{1}$$

you can sketch a right triangle with acute angle u, as shown in Figure 4.76. From this triangle, you can easily convert each expression to algebraic form.

a. $$\sin(\arccos 3x) = \sin u = \frac{\text{opp}}{\text{hyp}} = \sqrt{1-9x^2}, \quad 0 \le x \le \frac{1}{3}$$

b. $$\cot(\arccos 3x) = \cot u = \frac{\text{adj}}{\text{opp}} = \frac{3x}{\sqrt{1-9x^2}}, \quad 0 \le x < \frac{1}{3}$$

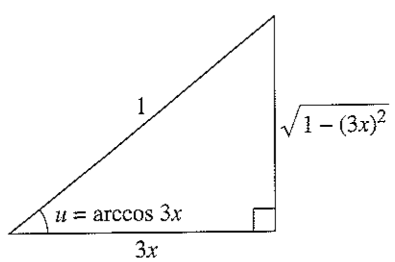


Figure 4.76

A similar argument can be made here for x-values lying in the interval $\left[-\frac{1}{3}, 0\right]$.

✓ *Checkpoint* Now try Exercise 47.

4.7 Exercises

Vocabulary Check

Fill in the blanks.

	Function	*Alternative Notation*	*Domain*	*Range*
1.	$y = \arcsin x$	______	______	$-\frac{\pi}{2} \le y \le \frac{\pi}{2}$
2.	______	$y = \cos^{-1} x$	$-1 \le x \le 1$	______
3.	$y = \arctan x$	______	______	______

In Exercises 1–7, find the exact value of each expression without using a calculator.

1. (a) $\arcsin \frac{1}{2}$ (b) $\arcsin 0$

2. (a) $\arccos \frac{1}{2}$ (b) $\arccos 0$

3. (a) $\arctan \frac{\sqrt{3}}{3}$ (b) $\arctan(-1)$

4. (a) $\cos^{-1}\left(-\frac{\sqrt{2}}{2}\right)$ (b) $\sin^{-1}\left(-\frac{\sqrt{2}}{2}\right)$

5. (a) $\arctan(-\sqrt{3})$ (b) $\arctan \sqrt{3}$

6. (a) $\arccos\left(-\frac{1}{2}\right)$ (b) $\arcsin \frac{\sqrt{2}}{2}$

7. (a) $\sin^{-1} \frac{\sqrt{3}}{2}$ (b) $\tan^{-1}\left(-\frac{\sqrt{3}}{3}\right)$

8. ***Numerical and Graphical Analysis*** Consider the function $y = \arcsin x$.

(a) Use a graphing utility to complete the table.

x	-1	-0.8	-0.6	-0.4	-0.2
y					

x	0	0.2	0.4	0.6	0.8	1
y						

(b) Plot the points from the table in part (a) and graph the function. (Do not use a graphing utility.)

(c) Use a graphing utility to graph the inverse sine function and compare the result with your hand-drawn graph in part (b).

(d) Determine any intercepts and symmetry of the graph.

9. ***Numerical and Graphical Analysis*** Consider the function $y = \arccos x$.

(a) Use a graphing utility to complete the table.

x	-1	-0.8	-0.6	-0.4	-0.2
y					

x	0	0.2	0.4	0.6	0.8	1
y						

(b) Plot the points from the table in part (a) and graph the function. (Do not use a graphing utility.)

(c) Use a graphing utility to graph the inverse cosine function and compare the result with your hand-drawn graph in part (b).

(d) Determine any intercepts and symmetry of the graph.

10. ***Numerical and Graphical Analysis*** Consider the function $y = \arctan x$.

(a) Use a graphing utility to complete the table.

x	-10	-8	-6	-4	-2
y					

x	0	2	4	6	8	10
y						

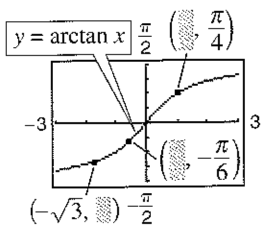
(b) Plot the points from the table in part (a) and graph the function. (Do not use a graphing utility.)

(c) Use a graphing utility to graph the inverse tangent function and compare the result with your hand-drawn graph in part (b).

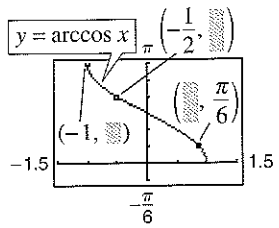
(d) Determine the horizontal asymptotes of the graph.

In Exercises 11 and 12, determine the missing coordinates of the points on the graph of the function.

11.



12.



In Exercises 13–24, use a calculator to approximate the value of the expression. Round your answer to two decimal places.

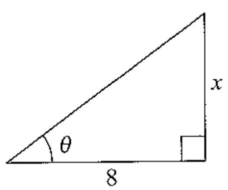
13. $\cos^{-1} 0.75$
14. $\sin^{-1} 0.56$
15. $\arcsin(-0.75)$
16. $\arccos(-0.7)$
17. $\arctan(-6)$
18. $\arctan(-18)$
19. $\sin^{-1} 0.19$
20. $\cos^{-1} 0.21$
21. $\arccos(-0.51)$
22. $\arcsin(-0.125)$
23. $\tan^{-1} 1.32$
24. $\tan^{-1} 5.9$

In Exercises 25 and 26, use a graphing utility to graph f, g, and $y = x$ in the same viewing window to verify geometrically that g is the inverse function of f. (Be sure to properly restrict the domain of f.)

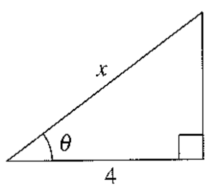
25. $f(x) = \tan x, \quad g(x) = \arctan x$
26. $f(x) = \sin x, \quad g(x) = \arcsin x$

In Exercises 27–30, use an inverse trigonometric function to write θ as a function of x.

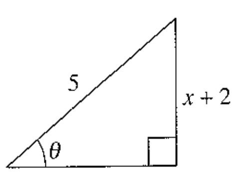
27.



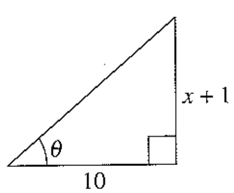
28.



29.



30.



In Exercises 31–38, use the properties of inverse functions to find the exact value of the expression.

31. $\sin(\arcsin 0.7)$
32. $\tan(\arctan 35)$
33. $\cos[\arccos(-0.3)]$
34. $\sin[\arcsin(-0.1)]$
35. $\arcsin(\sin 3\pi)$
36. $\arccos\left(\cos \frac{7\pi}{2}\right)$
37. $\tan^{-1}\left(\tan \frac{11\pi}{6}\right)$
38. $\sin^{-1}\left(\sin \frac{7\pi}{4}\right)$

In Exercises 39–46, find the exact value of the expression. Use a graphing utility to verify your result. (*Hint:* Make a sketch of a right triangle.)

39. $\sin\left(\arctan \frac{4}{3}\right)$
40. $\sec\left(\arcsin \frac{3}{5}\right)$
41. $\cos\left(\arcsin \frac{24}{25}\right)$
42. $\csc\left[\arctan\left(-\frac{12}{5}\right)\right]$
43. $\sec\left[\arctan\left(-\frac{3}{5}\right)\right]$
44. $\tan\left[\arcsin\left(-\frac{3}{4}\right)\right]$
45. $\sin\left[\arccos\left(-\frac{2}{3}\right)\right]$
46. $\cot\left(\arctan \frac{5}{8}\right)$

In Exercises 47–54, write an algebraic expression that is equivalent to the expression. (*Hint:* Sketch a right triangle, as demonstrated in Example 7.)

47. $\cot(\arctan x)$
48. $\sin(\arctan x)$
49. $\sin[\arccos(x + 2)]$
50. $\sec[\arcsin(x - 1)]$
51. $\tan\left(\arccos \frac{x}{5}\right)$
52. $\cot\left(\arctan \frac{4}{x}\right)$
53. $\csc\left(\arctan \frac{x}{\sqrt{7}}\right)$
54. $\cos\left(\arcsin \frac{x - h}{r}\right)$

In Exercises 55 and 56, use a graphing utility to graph f and g in the same viewing window to verify that the two functions are equal. Explain why they are equal. Identify any asymptotes of the graphs.

55. $f(x) = \sin(\arctan 2x), \quad g(x) = \dfrac{2x}{\sqrt{1 + 4x^2}}$

56. $f(x) = \tan\left(\arccos \dfrac{x}{2}\right), \quad g(x) = \dfrac{\sqrt{4 - x^2}}{x}$

In Exercises 57–60, complete the equation.

57. $\arctan \dfrac{14}{x} = \arcsin(\quad), \quad x > 0$

58. $\arcsin \dfrac{\sqrt{36 - x^2}}{6} = \arccos(\quad), \quad 0 \le x \le 6$

59. $\arccos \dfrac{3}{\sqrt{x^2 - 2x + 10}} = \arcsin(\quad)$

60. $\arccos \dfrac{x - 2}{2} = \arctan(\quad), \quad 2 < x < 4$

In Exercises 61–68, use a graphing utility to graph the function.

61. $y = 2 \arccos x$

62. $y = \arcsin \frac{x}{2}$

63. $f(x) = \arcsin(x - 2)$

64. $g(t) = \arccos(t + 2)$

65. $f(x) = \arctan 2x$

66. $f(x) = \pi + \arctan x$

67. $h(v) = \tan(\arccos v)$

68. $f(x) = \arccos \frac{x}{4}$

In Exercises 69 and 70, write the function in terms of the sine function by using the identity

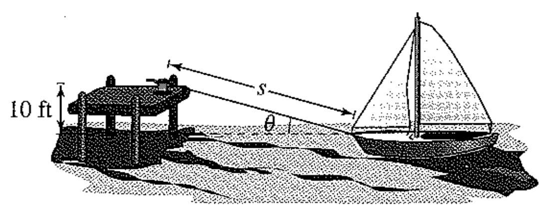
$$A \cos \omega t + B \sin \omega t = \sqrt{A^2 + B^2} \sin\left(\omega t + \arctan \frac{A}{B}\right).$$

Use a graphing utility to graph both forms of the function. What does the graph imply?

69. $f(t) = 3 \cos 2t + 3 \sin 2t$

70. $f(t) = 4 \cos \pi t + 3 \sin \pi t$

71. ***Docking a Boat*** A boat is pulled in by means of a winch located on a dock 10 feet above the deck of the boat (see figure). Let θ be the angle of elevation from the boat to the winch and let s be the length of the rope from the winch to the boat.



(a) Write θ as a function of s.

(b) Find θ when $s = 52$ feet and when $s = 26$ feet.

72. ***Granular Angle of Repose*** Different types of granular substances naturally settle at different angles when stored in cone-shaped piles. This angle θ is called the *angle of repose*. When rock salt is stored in a cone-shaped pile 11 feet high, the diameter of the pile's base is about 34 feet. (Source: Bulk-Store Structures, Inc.)

(a) Draw a diagram that gives a visual representation of the problem. Label all known and unknown quantities.

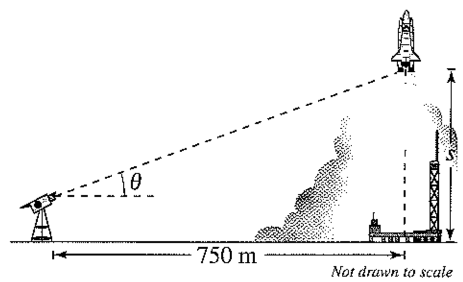
(b) Find the angle of repose for rock salt.

(c) How tall is a pile of rock salt that has a base diameter of 40 feet?

73. ***Photography*** A television camera at ground level is filming the lift-off of a space shuttle at a point 750 meters from the launch pad (see figure). Let θ be the angle of elevation to the shuttle and let s be the height of the shuttle.

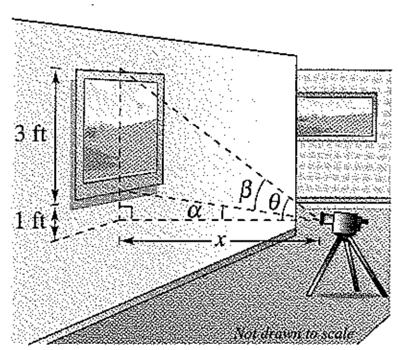
(a) Write θ as a function of s.

(b) Find θ when $s = 400$ meters and when $s = 1600$ meters.



74. ***Photography*** A photographer is taking a picture of a three-foot painting hung in an art gallery. The camera lens is 1 foot below the lower edge of the painting (see figure). The angle β subtended by the camera lens x feet from the painting is

$$\beta = \arctan \frac{3x}{x^2 + 4}, \quad x > 0.$$



(a) Use a graphing utility to graph β as a function of x.

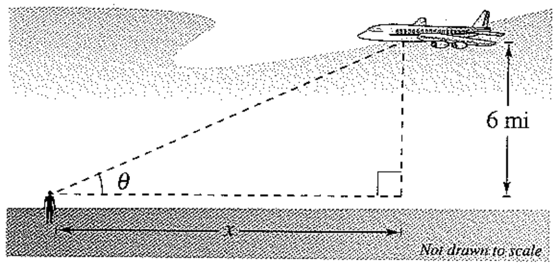
(b) Move the cursor along the graph to approximate the distance from the picture when β is maximum.

(c) Identify the asymptote of the graph and discuss its meaning in the context of the problem.

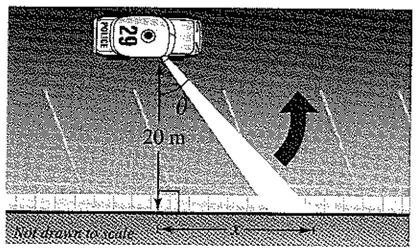
75. ***Angle of Elevation*** An airplane flies at an altitude of 6 miles toward a point directly over an observer. Consider θ and x as shown in the figure.

(a) Write θ as a function of x.

(b) Find θ when $x = 10$ miles and $x = 3$ miles.



76. ***Security Patrol*** A security car with its spotlight on is parked 20 meters from a long warehouse. Consider θ and x as shown in the figure.



(a) Write θ as a function of x.

(b) Find θ when $x = 5$ meters and when $x = 12$ meters.

Synthesis

True or False? **In Exercises 77 and 78, determine whether the statement is true or false. Justify your answer.**

77. $\sin \frac{5\pi}{6} = \frac{1}{2} \implies \arcsin \frac{1}{2} = \frac{5\pi}{6}$

78. $\arctan x = \dfrac{\arcsin x}{\arccos x}$

79. Define the inverse cotangent function by restricting the domain of the cotangent function to the interval $(0, \pi)$, and sketch the inverse function's graph.

80. Define the inverse secant function by restricting the domain of the secant function to the intervals $[0, \pi/2)$ and $(\pi/2, \pi]$, and sketch the inverse function's graph.

81. Define the inverse cosecant function by restricting the domain of the cosecant function to the intervals $[-\pi/2, 0)$ and $(0, \pi/2]$, and sketch the inverse function's graph.

82. Use the results of Exercises 79–81 to evaluate the following without using a calculator.

(a) $\operatorname{arcsec} \sqrt{2}$ (b) $\operatorname{arcsec} 1$

(c) $\operatorname{arccot}(-\sqrt{3})$ (d) $\operatorname{arccsc} 2$

Proof **In Exercises 83–85, prove the identity.**

83. $\arcsin(-x) = -\arcsin x$

84. $\arctan(-x) = -\arctan x$

85. $\arcsin x + \arccos x = \dfrac{\pi}{2}$

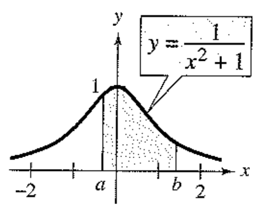
86. ***Area*** In calculus, it is shown that the area of the region bounded by the graphs of $y = 0$, $y = 1/(x^2 + 1)$, $x = a$, and $x = b$ is given by

Area = $\arctan b - \arctan a$

(see figure). Find the areas for each value of a and b.

(a) $a = 0, b = 1$ (b) $a = -1, b = 1$

(c) $a = 0, b = 3$ (d) $a = -1, b = 3$



Review

In Exercises 87–90, simplify the radical expression.

87. $\dfrac{4}{4\sqrt{2}}$ **88.** $\dfrac{2}{\sqrt{3}}$

89. $\dfrac{2\sqrt{3}}{6}$ **90.** $\dfrac{5\sqrt{5}}{2\sqrt{10}}$

In Exercises 91–94, sketch a right triangle corresponding to the trigonometric function of the acute angle θ. Use the Pythagorean Theorem to determine the third side and then find the other five trigonometric functions of θ.

91. $\sin \theta = \dfrac{5}{6}$ **92.** $\tan \theta = 2$

93. $\sin \theta = \dfrac{3}{4}$ **94.** $\sec \theta = 3$

4.8 Applications and Models

Applications Involving Right Triangles

In this section, the three angles of a right triangle are denoted by the letters A, B, and C (where C is the right angle), and the lengths of the sides opposite these angles by the letters a, b, and c (where c is the hypotenuse).

Example 1 Solving a Right Triangle

Solve the right triangle shown in Figure 4.77 for all unknown sides and angles.

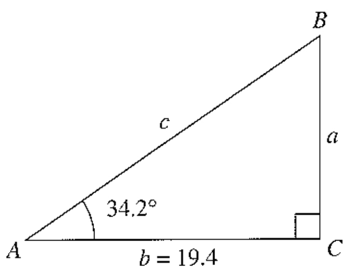


Figure 4.77

Solution

Because $C = 90°$, it follows that $A + B = 90°$ and $B = 90° - 34.2° = 55.8°$. To solve for a, use the fact that

$$\tan A = \frac{\text{opp}}{\text{adj}} = \frac{a}{b} \quad \Longrightarrow \quad a = b \tan A.$$

So, $a = 19.4 \tan 34.2° \approx 13.18$. Similarly, to solve for c, use the fact that

$$\cos A = \frac{\text{adj}}{\text{hyp}} = \frac{b}{c} \quad \Longrightarrow \quad c = \frac{b}{\cos A}.$$

So, $c = \dfrac{19.4}{\cos 34.2°} \approx 23.46$.

✓ ***Checkpoint*** Now try Exercise 1.

Recall from Section 4.3 that the term *angle of elevation* denotes the angle from the horizontal upward to an object and that the term *angle of depression* denotes the angle from the horizontal downward to an object. An angle of elevation and an angle of depression are shown in Figure 4.78.

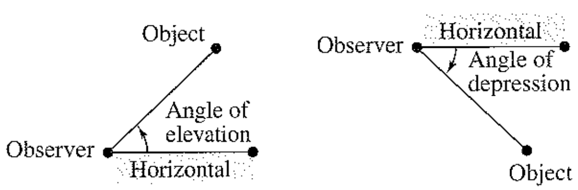


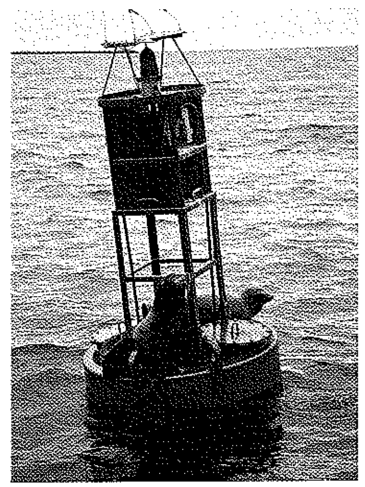
Figure 4.78

What you should learn

- Solve real-life problems involving right triangles.
- Solve real-life problems involving directional bearings.
- Solve real-life problems involving harmonic motion.

Why you should learn it

You can use trigonometric functions to model and solve real-life problems. For instance, Exercise 60 on page 330 shows you how a trigonometric function can be used to model the harmonic motion of a buoy.



Mary Kate Denny/PhotoEdit

Example 2 Finding a Side of a Right Triangle

A safety regulation states that the maximum angle of elevation for a rescue ladder is 72°. A fire department's longest ladder is 110 feet. What is the maximum safe rescue height?

Solution

A sketch is shown in Figure 4.79. From the equation $\sin A = a/c$, it follows that

$$a = c \sin A = 110 \sin 72° \approx 104.6.$$

So, the maximum safe rescue height is about 104.6 feet above the height of the fire truck.

Checkpoint Now try Exercise 17.

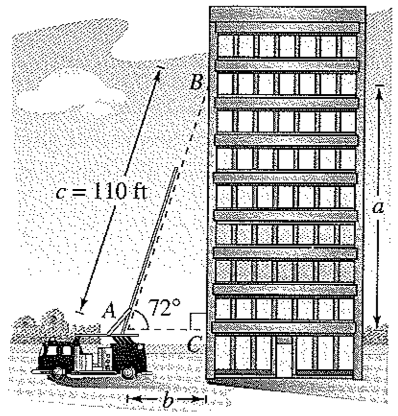


Figure 4.79

Example 3 Finding a Side of a Right Triangle

At a point 200 feet from the base of a building, the angle of elevation to the *bottom* of a smokestack is 35°, and the angle of elevation to the *top* is 53°, as shown in Figure 4.80. Find the height s of the smokestack alone.

Solution

This problem involves two right triangles. For the smaller right triangle, use the fact that $\tan 35° = a/200$ to conclude that the height of the building is

$$a = 200 \tan 35°.$$

Now, for the larger right triangle, use the equation

$$\tan 53° = \frac{a + s}{200}$$

to conclude that $s = 200 \tan 53° - a$. So, the height of the smokestack is

$$s = 200 \tan 53° - a = 200 \tan 53° - 200 \tan 35° \approx 125.4 \text{ feet.}$$

Checkpoint Now try Exercise 19.

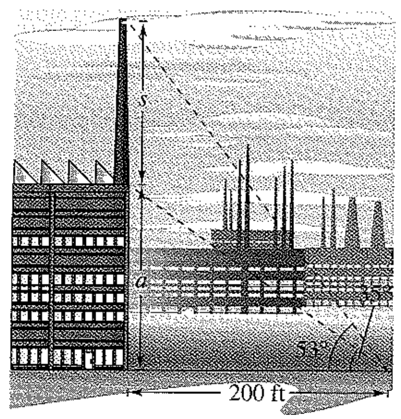


Figure 4.80

Example 4 Finding an Angle of Depression

A swimming pool is 20 meters long and 12 meters wide. The bottom of the pool is slanted so that the water depth is 1.3 meters at the shallow end and 4 meters at the deep end, as shown in Figure 4.81. Find the angle of depression of the bottom of the pool.

Solution

Using the tangent function, you see that

$$\tan A = \frac{\text{opp}}{\text{adj}} = \frac{2.7}{20} = 0.135.$$

So, the angle of depression is $A = \arctan 0.135 \approx 0.13419$ radian $\approx 7.69°$.

Checkpoint Now try Exercise 25.

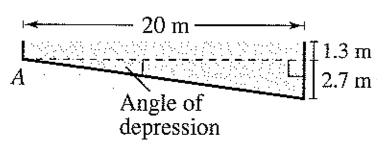


Figure 4.81

Trigonometry and Bearings

In surveying and navigation, directions are generally given in terms of **bearings.** A bearing measures the acute angle a path or line of sight makes with a fixed north–south line, as shown in Figure 4.82. For instance, the bearing of S 35° E in Figure 4.82(a) means 35 degrees east of south.

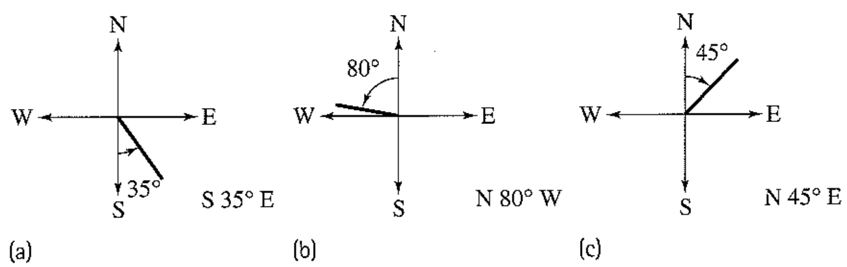


Figure 4.82

Example 5 Finding Directions in Terms of Bearings

A ship leaves port at noon and heads due west at 20 knots, or 20 nautical miles (nm) per hour. At 2 P.M. the ship changes course to N 54° W, as shown in Figure 4.83. Find the ship's bearing and distance from the port of departure at 3 P.M.

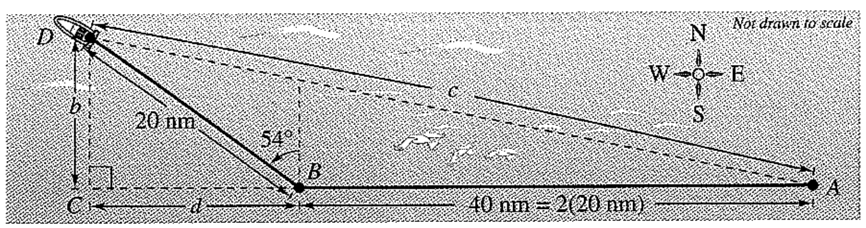


Figure 4.83

Solution

For triangle BCD, you have $B = 90° - 54° = 36°$. The two sides of this triangle can be determined to be

$$b = 20 \sin 36° \quad \text{and} \quad d = 20 \cos 36°.$$

In triangle ACD, you can find angle A as follows.

$$\tan A = \frac{b}{d + 40} = \frac{20 \sin 36°}{20 \cos 36° + 40} \approx 0.2092494$$

$$A \approx \arctan 0.2092494 \approx 0.2062732 \text{ radian} \approx 11.82°$$

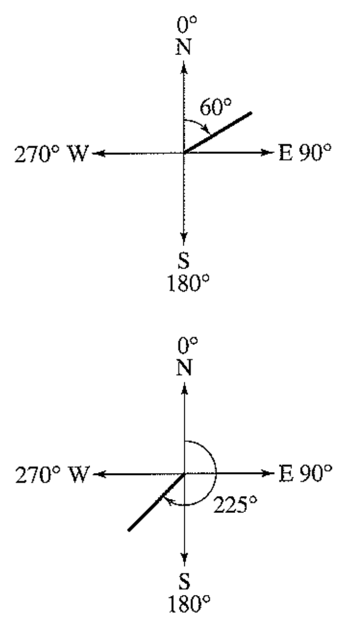
The angle with the north-south line is $90° - 11.82° = 78.18°$. So, the bearing of the ship is N 78.18° W. Finally, from triangle ACD, you have $\sin A = b/c$, which yields

$$c = \frac{b}{\sin A} = \frac{20 \sin 36°}{\sin 11.82°} \approx 57.4 \text{ nautical miles.} \qquad \text{Distance from port}$$

✓ ***Checkpoint*** Now try Exercise 31.

STUDY TIP

In *air navigation*, bearings are measured in degrees *clockwise* from north. Examples of air navigation bearings are shown below.



Harmonic Motion

The periodic nature of the trigonometric functions is useful for describing the motion of a point on an object that vibrates, oscillates, rotates, or is moved by wave motion.

For example, consider a ball that is bobbing up and down on the end of a spring, as shown in Figure 4.84. Suppose that 10 centimeters is the maximum distance the ball moves vertically upward or downward from its equilibrium (at-rest) position. Suppose further that the time it takes for the ball to move from its maximum displacement above zero to its maximum displacement below zero and back again is $t = 4$ seconds. Assuming the ideal conditions of perfect elasticity and no friction or air resistance, the ball would continue to move up and down in a uniform and regular manner.

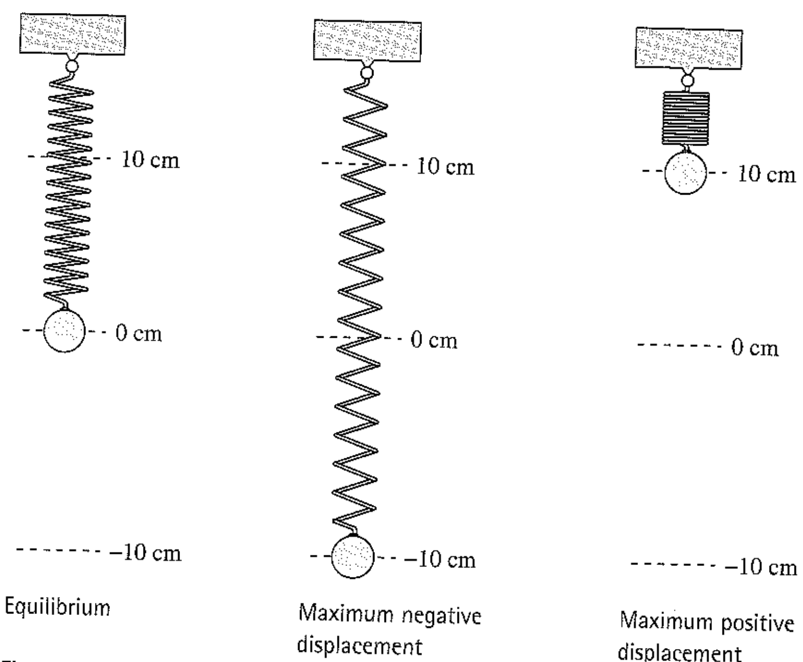


Figure 4.84

From this spring you can conclude that the period (time for one complete cycle) of the motion is

Period = 4 seconds

its amplitude (maximum displacement from equilibrium) is

Amplitude = 10 centimeters

and its **frequency** (number of cycles per second) is

$$\text{Frequency} = \frac{1}{4} \text{ cycle per second.}$$

Motion of this nature can be described by a sine or cosine function, and is called **simple harmonic motion.**

Definition of Simple Harmonic Motion

A point that moves on a coordinate line is said to be in **simple harmonic motion** if its distance d from the origin at time t is given by either

$$d = a \sin \omega t \quad \text{or} \quad d = a \cos \omega t$$

where a and ω are real numbers such that $\omega > 0$. The motion has amplitude $|a|$, period $2\pi/\omega$, and frequency $\omega/(2\pi)$.

Example 6 Simple Harmonic Motion

Write the equation for the simple harmonic motion of the ball illustrated in Figure 4.84, where the period is 4 seconds. What is the frequency of this motion?

Solution

Because the spring is at equilibrium ($d = 0$) when $t = 0$, you use the equation

$$d = a \sin \omega t.$$

Moreover, because the maximum displacement from zero is 10 and the period is 4, you have the following.

$$\text{Amplitude} = |a| = 10$$

$$\text{Period} = \frac{2\pi}{\omega} = 4 \quad \Longrightarrow \quad \omega = \frac{\pi}{2}$$

Consequently, the equation of motion is

$$d = 10 \sin \frac{\pi}{2} t.$$

Note that the choice of $a = 10$ or $a = -10$ depends on whether the ball initially moves up or down. The frequency is

$$\text{Frequency} = \frac{\omega}{2\pi}$$

$$= \frac{\pi/2}{2\pi}$$

$$= \frac{1}{4} \text{ cycle per second.}$$

✓ ***Checkpoint*** Now try Exercise 51.

One illustration of the relationship between sine waves and harmonic motion is the wave motion that results when a stone is dropped into a calm pool of water. The waves move outward in roughly the shape of sine (or cosine) waves, as shown in Figure 4.85. As an example, suppose you are fishing and your fishing bob is attached so that it does not move horizontally. As the waves move outward from the dropped stone, your fishing bob will move up and down in simple harmonic motion, as shown in Figure 4.86.

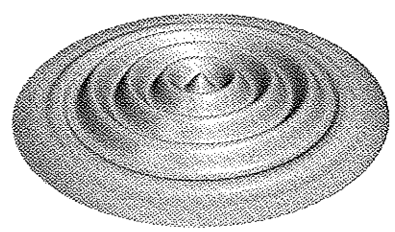


Figure 4.85

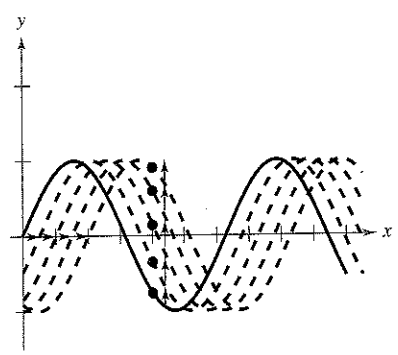


Figure 4.86

Example 7 Simple Harmonic Motion

Given the equation for simple harmonic motion

$$d = 6 \cos \frac{3\pi}{4} t$$

find (a) the maximum displacement, (b) the frequency, (c) the value of d when $t = 4$, and (d) the least positive value of t for which $d = 0$.

Algebraic Solution

The given equation has the form $d = a \cos \omega t$, with $a = 6$ and $\omega = 3\pi/4$.

a. The maximum displacement (from the point of equilibrium) is given by the amplitude. So, the maximum displacement is 6.

b. $$\text{Frequency} = \frac{\omega}{2\pi} = \frac{3\pi/4}{2\pi} = \frac{3}{8} \text{ cycle per unit of time}$$

c. $$d = 6 \cos\left[\frac{3\pi}{4}(4)\right] = 6 \cos 3\pi = 6(-1) = -6$$

d. To find the least positive value of t for which $d = 0$, solve the equation

$$d = 6 \cos \frac{3\pi}{4} t = 0.$$

First divide each side by 6 to obtain

$$\cos \frac{3\pi}{4} t = 0.$$

You know that $\cos t = 0$ when

$$t = \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}, \ldots.$$

Multiply these values by $4/(3\pi)$ to obtain

$$t = \frac{2}{3}, 2, \frac{10}{3}, \ldots.$$

So, the least positive value of t is $t = \frac{2}{3}$.

Checkpoint Now try Exercise 55.

Graphical Solution

Use a graphing utility set in *radian* mode to graph

$$y = 6 \cos \frac{3\pi}{4} x.$$

a. Use the *maximum* feature of the graphing utility to estimate that the maximum displacement from the point of equilibrium $y = 0$ is 6, as shown in Figure 4.87.

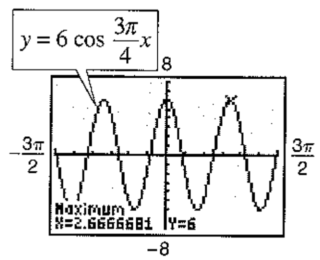


Figure 4.87

b. The period is the time for the graph to complete one cycle, which is $x \approx 2.667$. You can estimate the frequency as follows.

$$\text{Frequency} \approx \frac{1}{2.667} \approx 0.375 \text{ cycle per unit of time}$$

c. Use the *value* or *trace* feature to estimate that the value of y when $x = 4$ is $y = -6$, as shown in Figure 4.88.

d. Use the *zero* or *root* feature to estimate that the least positive value of x for which $y = 0$ is $x \approx 0.6667$, as shown in Figure 4.89.

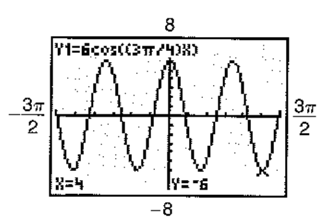


Figure 4.88

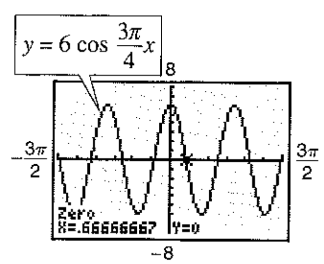


Figure 4.89

4.8 Exercises

Vocabulary Check

Fill in the blanks.

1. An angle that measures from the horizontal upward to an object is called the angle of ________ , whereas an angle that measures from the horizontal downward to an object is called the angle of ________ .
2. A ________ measures the acute angle a path or line of sight makes with a fixed north-south line.
3. A point that moves on a coordinate line is said to be in simple ________ if its distance from the origin at time t is given by either $d = a \sin \omega t$ or $d = a \cos \omega t$.

In Exercises 1–10, solve the right triangle shown in the figure. (Round your answers to two decimal places.)

1. $A = 20°,\ b = 10$
2. $B = 54°,\ c = 15$
3. $B = 71°,\ b = 24$
4. $A = 7.4°,\ a = 40.5$
5. $a = 6,\ b = 16$
6. $a = 25,\ c = 35$
7. $b = 16,\ c = 48$
8. $b = 1.32,\ c = 9.45$
9. $A = 12° 15',\ c = 430.5$
10. $B = 65° 12',\ a = 14.2$

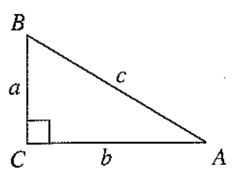


Figure for 1–10

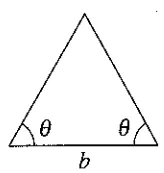


Figure for 11–14

In Exercises 11–14, find the altitude of the isosceles triangle shown in the figure. Round your answer to two decimal places.

11. $\theta = 52°,\ b = 4$ inches
12. $\theta = 18°,\ b = 10$ meters
13. $\theta = 41.6°,\ b = 14.2$ feet
14. $\theta = 72.94°,\ b = 3.36$ centimeters

15. ***Length*** A shadow of length L is created by a 60-foot silo when the sun is $\theta°$ above the horizon.
    (a) Draw a right triangle that gives a visual representation of the problem. Label the known and unknown quantities.
    (b) Write L as a function of θ.
    (c) Use a graphing utility to complete the table.

θ	10°	20°	30°	40°	50°
L					

    (d) The angle measure increases in equal increments in the table. Does the length of the shadow change in equal increments? Explain.

16. ***Length*** A shadow of length L is created by an 850-foot building when the sun is $\theta°$ above the horizon.
    (a) Draw a right triangle that gives a visual representation of the problem. Label the known and unknown quantities.
    (b) Write L as a function of θ.
    (c) Use a graphing utility to complete the table.

θ	10°	20°	30°	40°	50°
L					

    (d) The angle measure increases in equal increments in the table. Does the length of the shadow change in equal increments? Explain.

17. ***Height*** A ladder 20 feet long leans against the side of a house. The angle of elevation of the ladder is 80°. Find the height from the top of the ladder to the ground.
18. ***Height*** The angle of elevation from the base to the top of a waterslide is 13°. The slide extends horizontally 58.2 meters. Approximate the height of the waterslide.

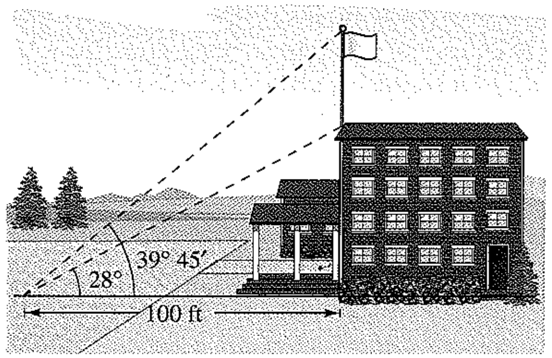
19. ***Height*** From a point 50 feet in front of a church, the angles of elevation to the base of the steeple and the top of the steeple are 35° and 47° 40′, respectively.

(a) Draw right triangles that give a visual representation of the problem. Label the known and unknown quantities.

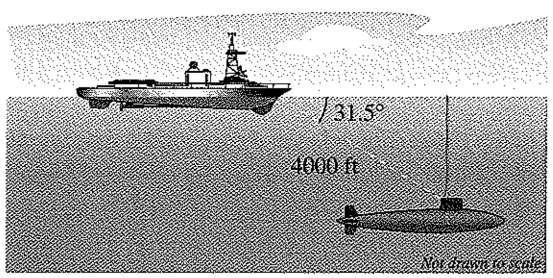
(b) Use a trigonometric function to write an equation involving the unknown quantity.

(c) Find the height of the steeple.

20. ***Height*** From a point 100 feet in front of a public library, the angles of elevation to the base of the flagpole and the top of the flagpole are 28° and 39° 45′, respectively. The flagpole is mounted on the front of the library's roof. Find the height of the flagpole.



21. ***Depth*** The sonar of a navy cruiser detects a submarine that is 4000 feet from the cruiser. The angle between the water level and the submarine is 31.5°. How deep is the submarine?



22. ***Height*** A 100-foot line is attached to a kite. When the kite has pulled the line taut, the angle of elevation to the kite is approximately 50°. Approximate the height of the kite.

23. ***Angle of Elevation*** An engineer erects a 75-foot vertical cellular-phone tower. Find the angle of elevation to the top of the tower from a point on level ground 95 feet from its base.

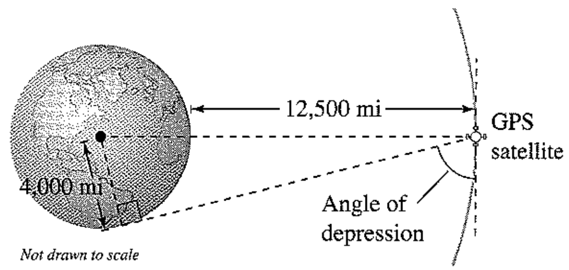
24. ***Angle of Elevation*** The height of an outdoor basketball backboard is $12\frac{1}{2}$ feet, and the backboard casts a shadow $17\frac{1}{3}$ feet long.

(a) Draw a right triangle that gives a visual representation of the problem. Label the known and unknown quantities.

(b) Use a trigonometric function to write an equation involving the unknown quantity.

(c) Find the angle of elevation of the sun.

25. ***Angle of Depression*** A Global Positioning System satellite orbits 12,500 miles above Earth's surface. Find the angle of depression from the satellite to the horizon. Assume the radius of Earth is 4000 miles.

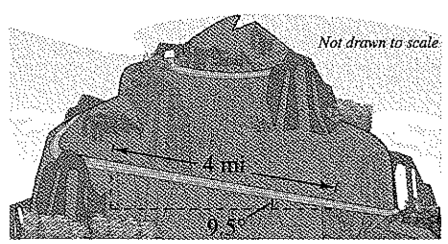


26. ***Angle of Depression*** Find the angle of depression from the top of a lighthouse 250 feet above water level to the water line of a ship $2\frac{1}{2}$ miles offshore.

27. ***Airplane Ascent*** When an airplane leaves the runway, its angle of climb is 18° and its speed is 275 feet per second. Find the plane's altitude after 1 minute.

28. ***Airplane Ascent*** How long will it take the plane in Exercise 27 to climb to an altitude of 10,000 feet? 16,000 feet?

29. ***Mountain Descent*** A sign on the roadway at the top of a mountain indicates that for the next 4 miles the grade is 9.5° (see figure). Find the change in elevation for a car descending the mountain.

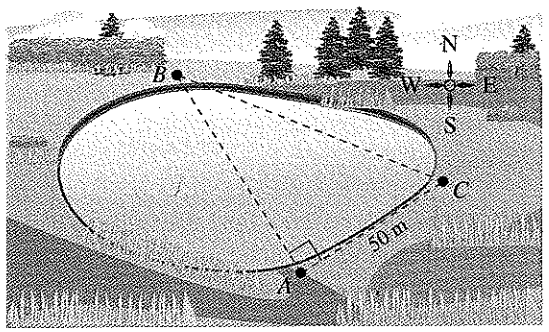


30. ***Ski Slope*** A ski slope on a mountain has an angle of elevation of 25.2°. The vertical height of the slope is 1808 feet. How long is the slope?

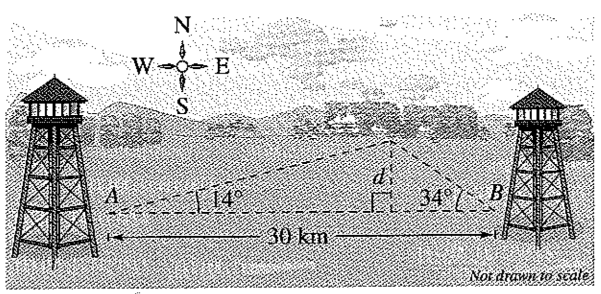
31. ***Navigation*** A ship leaves port at noon and has a bearing of S 29° W. The ship sails at 20 knots. How many nautical miles south and how many nautical miles west will the ship have traveled by 6:00 P.M.?

32. ***Navigation*** An airplane flying at 600 miles per hour has a bearing of 52°. After flying 1.5 hours, how far north and how far east has the plane traveled from its point of departure?

33. ***Surveying*** A surveyor wants to find the distance across a swamp. The bearing from A to B is N 32° W. The surveyor walks 50 meters from A, and at the point C the bearing to B is N 68° W. Find (a) the bearing from A to C and (b) the distance from A to B.



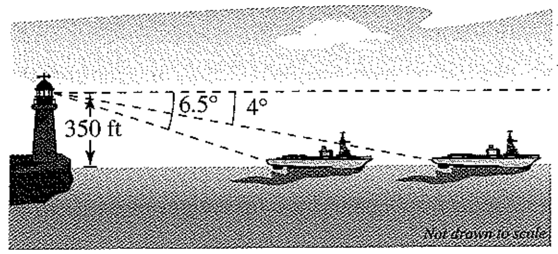
34. ***Location of a Fire*** Two fire towers are 30 kilometers apart, where tower A is due west of tower B. A fire is spotted from the towers, and the bearings from A and B are E 14° N and W 34° N, respectively. Find the distance d of the fire from the line segment AB.



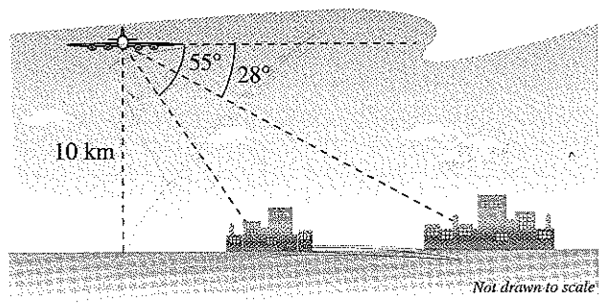
35. ***Navigation*** A ship is 45 miles east and 30 miles south of port. The captain wants to sail directly to port. What bearing should be taken?

36. ***Navigation*** A plane is 160 miles north and 85 miles east of an airport. The pilot wants to fly directly to the airport. What bearing should be taken?

37. ***Distance*** An observer in a lighthouse 350 feet above sea level observes two ships directly offshore. The angles of depression to the ships are 4° and 6.5° (see figure). How far apart are the ships?



38. ***Distance*** A passenger in an airplane flying at an altitude of 10 kilometers sees two towns directly to the east of the plane. The angles of depression to the towns are 28° and 55° (see figure). How far apart are the towns?



39. ***Altitude*** A plane is observed approaching your home and you assume its speed is 550 miles per hour. The angle of elevation to the plane is 16° at one time and 57° one minute later. Approximate the altitude of the plane.

40. ***Height*** While traveling across flat land, you notice a mountain directly in front of you. The angle of elevation to the peak is 2.5°. After you drive 18 miles closer to the mountain, the angle of elevation is 10°. Approximate the height of the mountain.

Geometry **In Exercises 41 and 42, find the angle α between two nonvertical lines L_1 and L_2. The angle α satisfies the equation**

$$\tan \alpha = \left| \frac{m_2 - m_1}{1 + m_2 m_1} \right|$$

where m_1 and m_2 are the slopes of L_1 and L_2, respectively. (Assume $m_1 m_2 \neq -1$.)

41. L_1: $3x - 2y = 5$
L_2: $x + y = 1$

42. L_1: $2x + y = 8$
L_2: $x - 5y = -4$

43. ***Geometry*** Determine the angle between the diagonal of a cube and the diagonal of its base, as shown in the figure.

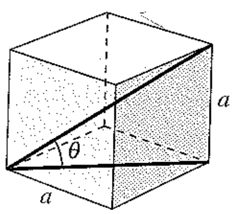


Figure for 43

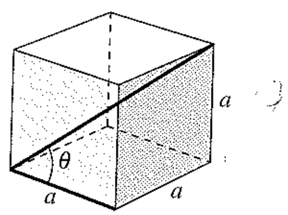
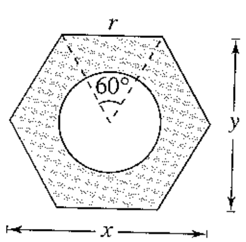


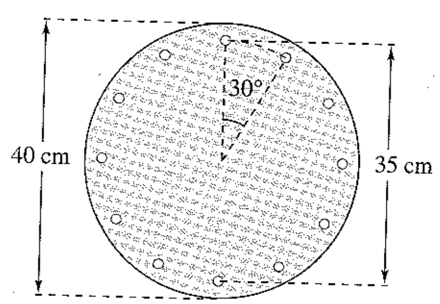
Figure for 44

44. ***Geometry*** Determine the angle between the diagonal of a cube and its edge, as shown in the figure.

45. ***Hardware*** Write the distance y across the flat sides of a hexagonal nut as a function of r, as shown in the figure.



46. ***Hardware*** The figure shows a circular piece of sheet metal of diameter 40 centimeters. The sheet contains 12 equally spaced bolt holes. Determine the straight-line distance between the centers of two consecutive bolt holes.

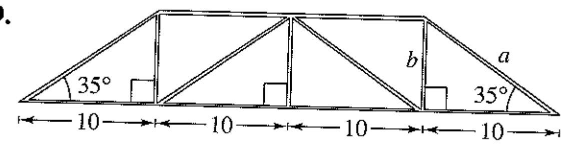


47. ***Geometry*** A regular pentagon (a pentagon with congruent sides and angles) is inscribed in a circle of radius 25 inches. Find the length of the sides of the pentagon.

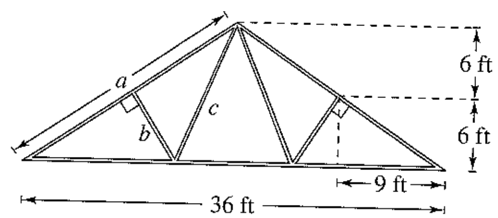
48. ***Geometry*** A regular hexagon (a hexagon with congruent sides and angles) is inscribed in a circle of radius 25 inches. Find the length of the sides of the hexagon.

Trusses **In Exercises 49 and 50, find the lengths of all the unknown members of the truss.**

49.



50.



Harmonic Motion **In Exercises 51–54, find a model for simple harmonic motion satisfying the specified conditions.**

	Displacement ($t = 0$)	*Amplitude*	*Period*
51.	0	8 centimeters	2 seconds
52.	0	3 meters	6 seconds
53.	3 inches	3 inches	1.5 seconds
54.	2 feet	2 feet	10 seconds

Harmonic Motion **In Exercises 55–58, for the simple harmonic motion described by the trigonometric function, find (a) the maximum displacement, (b) the frequency, (c) the value of d when $t = 5$, and (d) the least positive value of t for which $d = 0$. Use a graphing utility to verify your results.**

55. $d = 4 \cos 8\pi t$

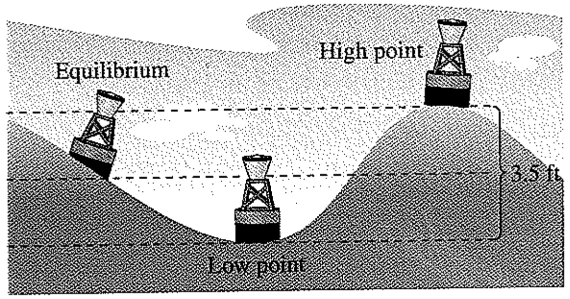
56. $d = \frac{1}{2} \cos 20\pi t$

57. $d = \frac{1}{16} \sin 140\pi t$

58. $d = \frac{1}{64} \sin 792\pi t$

59. ***Tuning Fork*** A point on the end of a tuning fork moves in the simple harmonic motion described by $d = a \sin \omega t$. Find ω given that the tuning fork for middle C has a frequency of 264 vibrations per second.

60. ***Wave Motion*** A buoy oscillates in simple harmonic motion as waves go past. At a given time it is noted that the buoy moves a total of 3.5 feet from its low point to its high point (see figure), and that it returns to its high point every 10 seconds. Write an equation that describes the motion of the buoy if it is at its high point at time $t = 0$.



61. ***Springs*** A ball that is bobbing up and down on the end of a spring has a maximum displacement of 3 inches. Its motion (in ideal conditions) is modeled by

$$y = \frac{1}{4}\cos 16t, \quad t > 0$$

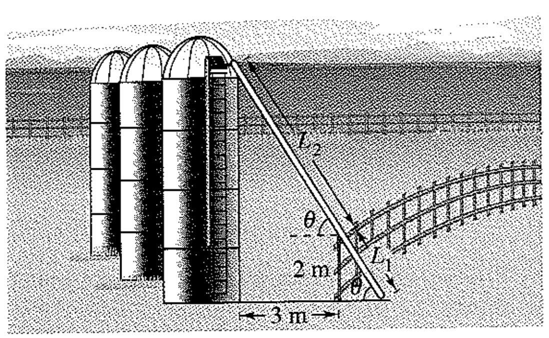
where y is measured in feet and t is the time in seconds.

(a) Use a graphing utility to graph the function.

(b) What is the period of the oscillations?

(c) Determine the first time the ball passes the point of equilibrium $(y = 0)$.

62. ***Numerical and Graphical Analysis*** A two-meter-high fence is 3 meters from the side of a grain storage bin. A grain elevator must reach from ground level outside the fence to the storage bin (see figure). The objective is to determine the shortest elevator that meets the constraints.



(a) Complete four rows of the table.

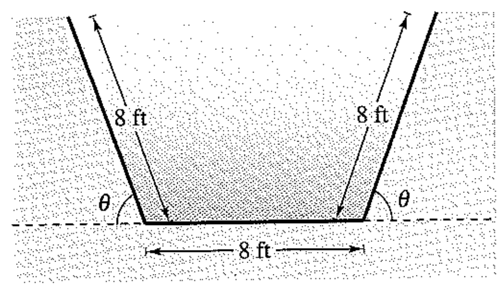
θ	L_1	L_2	$L_1 + L_2$
0.1	$\frac{2}{\sin 0.1}$	$\frac{3}{\cos 0.1}$	23.0
0.2	$\frac{2}{\sin 0.2}$	$\frac{3}{\cos 0.2}$	13.1

(b) Use the *table* feature of a graphing utility to generate additional rows of the table. Use the table to estimate the minimum length of the elevator.

(c) Write the length $L_1 + L_2$ as a function of θ.

(d) Use a graphing utility to graph the function. Use the graph to estimate the minimum length. How does your estimate compare with that in part (b)?

63. ***Numerical and Graphical Analysis*** The cross sections of an irrigation canal are isosceles trapezoids, where the length of three of the sides is 8 feet (see figure). The objective is to find the angle θ that maximizes the area of the cross sections. [*Hint:* The area of a trapezoid is given by $(h/2)(b_1 + b_2)$.]



(a) Complete seven rows of the table.

Base 1	*Base 2*	*Altitude*	*Area*
8	$8 + 16\cos 10°$	$8\sin 10°$	22.1
8	$8 + 16\cos 20°$	$8\sin 20°$	42.5

(b) Use the *table* feature of a graphing utility to generate additional rows of the table. Use the table to estimate the maximum cross-sectional area.

(c) Write the area A as a function of θ.

(d) Use a graphing utility to graph the function. Use the graph to estimate the maximum cross-sectional area. How does your estimate compare with that in part (b)?

64. ***Data Analysis*** The times S of sunset (Greenwich Mean Time) at 40° north latitude on the 15th of each month are: 1(16:59), 2(17:35), 3(18:06), 4(18:38), 5(19:08), 6(19:30), 7(19:28), 8(18:57), 9(18:09), 10(17:21), 11(16:44), 12(16:36). The month is represented by t, with $t = 1$ corresponding to January. A model (in which minutes have been converted to the decimal parts of an hour) for this data is given by

$$S(t) = 18.09 + 1.41 \sin\left(\frac{\pi t}{6} + 4.60\right).$$

(a) Use a graphing utility to graph the data points and the model in the same viewing window.

(b) What is the period of the model? Is it what you expected? Explain.

(c) What is the amplitude of the model? What does it represent in the context of the problem? Explain.

65. ***Data Analysis*** The table shows the average sales S (in millions of dollars) of an outerwear manufacturer for each month t, where $t = 1$ represents January.

Month, t	Sales, S
1	13.46
2	11.15
3	8.00
4	4.85
5	2.54
6	1.70
7	2.54
8	4.85
9	8.00
10	11.15
11	13.46
12	14.30

(a) Create a scatter plot of the data.

(b) Find a trigonometric model that fits the data. Graph the model on your scatter plot. How well does the model fit?

(c) What is the period of the model? Do you think it is reasonable given the context? Explain your reasoning.

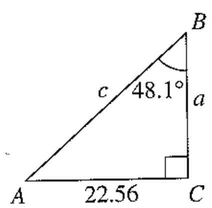
(d) Interpret the meaning of the model's amplitude in the context of the problem.

66. ***Writing*** Is it true that N 24° E means 24 degrees north of east? Explain.

Synthesis

True or False? **In Exercises 67 and 68, determine whether the statement is true or false. Justify your answer.**

67. In the right triangle shown below, $a = \dfrac{22.56}{\tan 41.9°}$.



68. For the harmonic motion of a ball bobbing up and down on the end of a spring, one period can be described as the length of one coil of the spring.

Review

In Exercises 69–72, write the standard form of the equation of the line that has the specified characteristics.

69. $m = 4$, passes through $(-1, 2)$

70. $m = -\frac{1}{2}$, passes through $\left(\frac{1}{3}, 0\right)$

71. Passes through $(-2, 6)$ and $(3, 2)$

72. Passes through $\left(\frac{1}{4}, -\frac{2}{3}\right)$ and $\left(-\frac{1}{2}, \frac{1}{3}\right)$

In Exercises 73–80, find the domain of the function.

73. $f(x) = 3x + 8$

74. $f(x) = -x^2 - 1$

75. $g(x) = \sqrt[3]{x + 2}$

76. $g(x) = \sqrt{7 - x}$

77. $h(x) = \dfrac{2}{x^2 - 2x}$

78. $h(x) = \dfrac{x}{3x + 5}$

79. $f(x) = 4e^{-x}$

80. $f(x) = \ln(x - 2)$

In Exercises 81–84, solve the equation. Round your answer to three decimal places.

81. $e^{2x} = 54$

82. $\dfrac{300}{1 + e^{-x}} = 100$

83. $\ln(x^2 + 1) = 3.2$

84. $\log_8 x + \log_8(x - 1) = \frac{1}{3}$

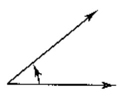
4 Chapter Summary

What did you learn?

	Review Exercises
Section 4.1	
☐ Describe angles.	1, 2
☐ Use radian and degree measure, and convert between radian and degree measure.	3–34
☐ Use angles to model and solve real-life problems.	35–40
Section 4.2	
☐ Identify a unit circle and describe its relationship to real numbers.	41–44
☐ Evaluate trigonometric functions using the unit circle.	45–48
☐ Use domain and period to evaluate sine and cosine functions.	49–52
☐ Use a calculator to evaluate trigonometric functions.	53–56
Section 4.3	
☐ Evaluate trigonometric functions of acute angles.	57–60
☐ Use the fundamental trigonometric identities.	61, 62
☐ Use a calculator to evaluate trigonometric functions.	63–66
☐ Use trigonometric functions to model and solve real-life problems.	67, 68
Section 4.4	
☐ Evaluate trigonometric functions of any angle.	69–78
☐ Use reference angles to evaluate trigonometric functions.	79–90
☐ Evaluate trigonometric functions of real numbers.	91–94
Section 4.5	
☐ Sketch the graphs of basic sine and cosine functions.	95–98
☐ Use amplitude and period to sketch the graphs of sine and cosine functions.	99–108
☐ Sketch translations of graphs of sine and cosine functions.	109–118
☐ Use sine and cosine functions to model real-life data.	119, 120
Section 4.6	
☐ Sketch the graphs of tangent and cotangent functions.	121–130
☐ Sketch the graphs of secant and cosecant functions.	131–140
☐ Sketch the graphs of damped trigonometric functions.	141–144
Section 4.7	
☐ Evaluate inverse trigonometric functions.	145–158
☐ Evaluate compositions of trigonometric functions.	159–162
Section 4.8	
☐ Solve real-life problems involving right triangles.	163–165
☐ Solve real-life problems involving directional bearings.	166
☐ Solve real-life problems involving harmonic motion.	167, 168

4 Review Exercises

4.1 In Exercises 1 and 2, estimate the angle to the nearest one-half radian.

1. 

2. 

In Exercises 3–6, (a) sketch the angle in standard position, (b) determine the quadrant in which the angle lies, and (c) list one positive and one negative coterminal angle.

3. $\dfrac{\pi}{16}$
4. $\dfrac{40\pi}{47}$
5. $-\dfrac{9\pi}{15}$
6. $-\dfrac{11\pi}{3}$

In Exercises 7–10, find (if possible) the complement and supplement of the angle.

7. $\dfrac{\pi}{8}$
8. $\dfrac{13\pi}{18}$
9. $\dfrac{3\pi}{10}$
10. $\dfrac{2\pi}{21}$

In Exercises 11–14, (a) sketch the angle in standard position, (b) determine the quadrant in which the angle lies, and (c) list one positive and one negative coterminal angle.

11. $35°$
12. $190°$
13. $-110°$
14. $-420°$

In Exercises 15–18, find (if possible) the complement and supplement of each angle.

15. $8°$
16. $94°$
17. $171°$
18. $49°$

In Exercises 19–22, use the angle conversion capabilities of a graphing utility to convert the angle measure to decimal degree form. Round your answer to three decimal places.

19. $135°\,16'\,45''$
20. $-234°\,40''$
21. $5°\,22'\,53''$
22. $280°\,8'\,50''$

In Exercises 23–26, use the angle-conversion capabilities of a graphing utility to convert the angle measure to D°M′S″ form.

23. $135.29°$
24. $25.8°$
25. $-85.36°$
26. $-327.93°$

In Exercises 27–30, convert the angle measure from degrees to radians. Round your answer to four decimal places.

27. $480°$
28. $-16.5°$
29. $-33°$
30. $84°$

In Exercises 31–34, convert the angle measure from radians to degrees. Round your answer to three decimal places.

31. $\dfrac{5\pi}{7}$
32. $-\dfrac{3\pi}{5}$
33. -3.5
34. 1.55

35. Find the radian measure of the central angle of a circle with a radius of 12 feet that intercepts an arc of length 25 feet.

36. Find the radian measure of the central angle of a circle with a radius of 60 inches that intercepts an arc of length 245 inches.

37. Find the length of the arc on a circle with a radius of 20 meters intercepted by a central angle of 138°.

38. Find the length of the arc on a circle with a radius of 15 centimeters intercepted by a central angle of 60°.

39. ***Music*** The radius of a compact disc is 6 centimeters. Find the linear speed of a point on the circumference of the disc if it is rotating at a speed of 500 revolutions per minute.

40. ***Angular Speed*** A car is moving at a rate of 28 miles per hour, and the diameter of its wheels is about $2\frac{1}{3}$ feet.
    (a) Find the number of revolutions per minute the wheels are rotating.
    (b) Find the angular speed of the wheels in radians per minute.

4.2 In Exercises 41–44, find the point (x, y) on the unit circle that corresponds to the real number t.

41. $t = \frac{7\pi}{4}$
42. $t = \frac{3\pi}{4}$
43. $t = \frac{5\pi}{6}$
44. $t = -\frac{4\pi}{3}$

In Exercises 45–48, evaluate (if possible) the six trigonometric functions of the real number.

45. $t = \frac{7\pi}{6}$
46. $t = \frac{\pi}{4}$
47. $t = -\frac{2\pi}{3}$
48. $t = \pi$

In Exercises 49–52, evaluate the trigonometric function using its period as an aid.

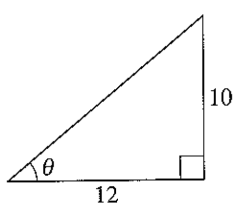
49. $\sin \frac{11\pi}{4}$
50. $\cos 4\pi$
51. $\sin\left(-\frac{17\pi}{6}\right)$
52. $\cos\left(-\frac{13\pi}{3}\right)$

In Exercises 53–56, use a calculator to evaluate the expression. Round your answer to four decimal places.

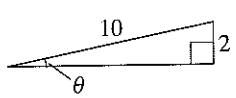
53. $\cot 2.3$
54. $\sec 4.5$
55. $\cos \frac{5\pi}{3}$
56. $\tan\left(-\frac{11\pi}{6}\right)$

4.3 In Exercises 57–60, find the exact values of the six trigonometric functions of the angle θ shown in the figure.

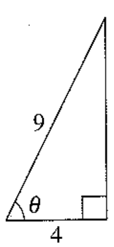
57.



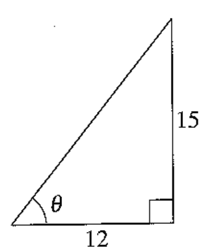
58.



59.



60.



In Exercises 61 and 62, use trigonometric identities to transform one side of the equation into the other.

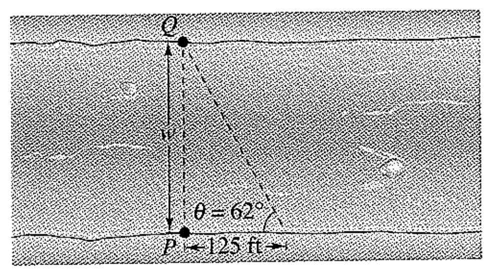
61. $\csc \theta \tan \theta = \sec \theta$

62. $\frac{\cot \theta + \tan \theta}{\cot \theta} = \sec^2 \theta$

In Exercises 63–66, use a calculator to evaluate each function. Round your answers to four decimal places.

63. (a) $\cos 84°$ (b) $\sin 6°$
64. (a) $\csc 52° 12'$ (b) $\sec 54° 7'$
65. (a) $\cos \frac{\pi}{4}$ (b) $\sec \frac{\pi}{4}$
66. (a) $\tan \frac{3\pi}{20}$ (b) $\cot \frac{3\pi}{20}$

67. ***Width*** An engineer is trying to determine the width of a river. From point P, the engineer walks downstream 125 feet and sights to point Q. From this sighting, it is determined that $\theta = 62°$. How wide is the river?



68. ***Height*** An escalator 152 feet in length rises to a platform and makes a 30° angle with the ground.
(a) Draw a right triangle that gives a visual representation of the problem. Show the known quantities of the triangle and use a variable to indicate the height of the platform above the ground.
(b) Use a trigonometric function to write an equation involving the unknown quantity.
(c) Find the height of the platform above the ground.

4.4 In Exercises 69–74, the point is on the terminal side of an angle in standard position. Determine the exact values of the six trigonometric functions of the angle.

69. $(12, 16)$
70. $(-4, -6)$
71. $(-7, 2)$
72. $(4, -8)$
73. $(2, 5)$
74. $(-9, 3)$

In Exercises 75–78, find the values of the six trigonometric functions of θ satisfying the given conditions.

75. $\sec \theta = \frac{6}{5}$, $\tan \theta < 0$

76. $\tan \theta = -\frac{12}{5}$, $\sin \theta > 0$

77. $\sin \theta = \frac{3}{8}$, $\cos \theta < 0$

78. $\cos \theta = -\frac{2}{5}$, $\sin \theta > 0$

In Exercises 79–82, find the reference angle θ' and sketch θ and θ' in standard position.

79. $\theta = 264°$

80. $\theta = 635°$

81. $\theta = -\dfrac{6\pi}{5}$

82. $\theta = \dfrac{17\pi}{3}$

In Exercises 83–90, evaluate the sine, cosine, and tangent of the angle without using a calculator.

83. $240°$

84. $315°$

85. $-210°$

86. $-315°$

87. $-\dfrac{9\pi}{4}$

88. $\dfrac{11\pi}{6}$

89. $\dfrac{\pi}{2}$

90. $-\dfrac{\pi}{3}$

In Exercises 91–94, use a calculator to evaluate the trigonometric function. Round to four decimal places.

91. $\tan 33°$

92. $\csc 105°$

93. $\sec \dfrac{12\pi}{5}$

94. $\sin\left(-\dfrac{\pi}{9}\right)$

4.5 In Exercises 95–98, sketch the graph of the function.

95. $f(x) = 3 \sin x$

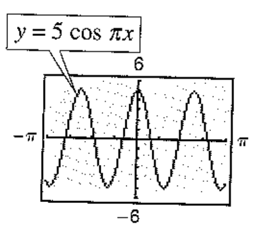
96. $f(x) = 2 \cos x$

97. $f(x) = \frac{1}{4} \cos x$

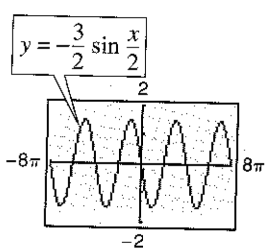
98. $f(x) = \frac{7}{2} \sin x$

In Exercises 99–102, find the period and amplitude.

99.



100.



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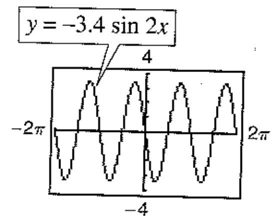

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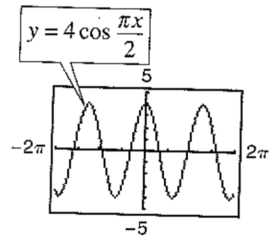


In Exercises 103–114, sketch the graph of the function. (Include two full periods.)

103. $f(x) = 3 \cos 2\pi x$

104. $f(x) = -2 \sin \pi x$

105. $f(x) = 5 \sin \dfrac{2x}{5}$

106. $f(x) = 8 \cos\left(-\dfrac{x}{4}\right)$

107. $f(x) = -\dfrac{5}{2} \cos \dfrac{x}{4}$

108. $f(x) = -\dfrac{1}{2} \sin \dfrac{\pi x}{4}$

109. $f(x) = \frac{5}{2} \sin(x - \pi)$

110. $f(x) = 3 \cos(x + \pi)$

111. $f(x) = 2 - \cos \dfrac{\pi x}{2}$

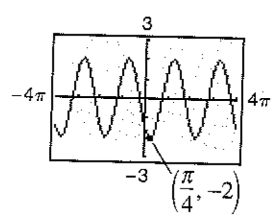
112. $f(x) = \frac{1}{2} \sin \pi x - 3$

113. $f(x) = -3 \cos\left(\dfrac{x}{2} - \dfrac{\pi}{4}\right)$

114. $f(x) = 4 - 2 \cos(4x + \pi)$

Graphical Reasoning **In Exercises 115–118, find a, b, and c for the function $f(x) = a \cos(bx - c)$ such that the graph of f matches the graph shown.**

115.



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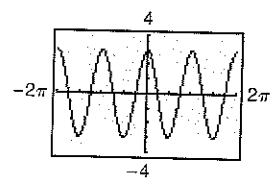

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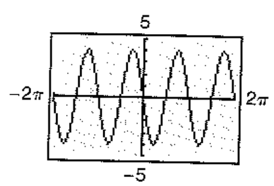

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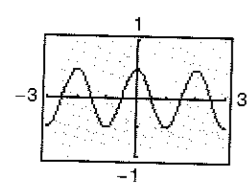


Sales **In Exercises 119 and 120, use a graphing utility to graph the sales function over 1 year, where S is the sales (in thousands of units) and t is the time (in months), with $t = 1$ corresponding to January. Determine the months of maximum and minimum sales.**

119. $S = 48.4 - 6.1 \cos \frac{\pi t}{6}$

120. $S = 56.25 + 9.50 \sin \frac{\pi t}{6}$

4.6

In Exercises 121–140, sketch the graph of the function. (Include two full periods.)

121. $f(x) = -\tan \frac{\pi x}{4}$ **122.** $f(x) = 4 \tan \pi x$

123. $f(x) = \frac{1}{4} \tan \frac{\pi x}{2}$ **124.** $f(x) = \tan\left(x + \frac{\pi}{4}\right)$

125. $f(x) = -\frac{1}{4} \tan\left(x - \frac{\pi}{2}\right)$ **126.** $f(x) = 2 + 2 \tan \frac{x}{3}$

127. $f(x) = 3 \cot \frac{x}{2}$ **128.** $f(x) = \frac{1}{2} \cot \frac{\pi x}{2}$

129. $f(x) = \frac{1}{2} \cot\left(x - \frac{\pi}{2}\right)$

130. $f(x) = 4 \cot\left(x + \frac{\pi}{4}\right)$

131. $f(x) = \frac{1}{4} \sec x$ **132.** $f(x) = \frac{1}{2} \csc x$

133. $f(x) = \frac{1}{4} \csc 2x$ **134.** $f(x) = \frac{1}{2} \sec 2\pi x$

135. $f(x) = \sec\left(x - \frac{\pi}{4}\right)$

136. $f(x) = \frac{1}{2} \csc(2x + \pi)$

137. $f(x) = 2 \sec(x - \pi)$

138. $f(x) = -2 \csc(x - \pi)$

139. $f(x) = \csc\left(3x - \frac{\pi}{2}\right)$

140. $f(x) = 3 \csc\left(2x + \frac{\pi}{4}\right)$

In Exercises 141–144, use a graphing utility to graph the function and the damping factor of the function in the same viewing window. Describe the behavior of the function as x increases without bound.

141. $f(x) = e^x \sin 2x$ **142.** $f(x) = 2x \cos x$

143. $f(x) = e^x \cos x$ **144.** $f(x) = x \sin \pi x$

4.7

In Exercises 145–148, find the value of each expression without using a calculator.

145. (a) $\arcsin 1$ (b) $\arcsin 4$

146. (a) $\arcsin\left(-\frac{\sqrt{2}}{2}\right)$ (b) $\arcsin \frac{\sqrt{3}}{2}$

147. (a) $\cos^{-1} \frac{\sqrt{2}}{2}$ (b) $\cos^{-1}\left(-\frac{\sqrt{3}}{2}\right)$

148. (a) $\tan^{-1}(-\sqrt{3})$ (b) $\tan^{-1} 1$

In Exercises 149–156, use a calculator to approximate the value of the expression. Round your answer to two decimal places.

149. $\arccos 0.42$ **150.** $\arcsin 0.63$

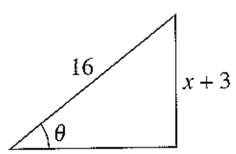
151. $\sin^{-1}(-0.94)$ **152.** $\cos^{-1}(-0.12)$

153. $\arctan(-12)$ **154.** $\arctan 21$

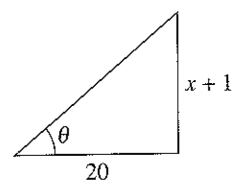
155. $\tan^{-1} 0.81$ **156.** $\tan^{-1} 6.4$

In Exercises 157 and 158, use an inverse trigonometric function to write θ as a function of x.

157.



158.



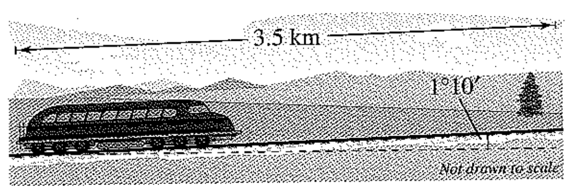
In Exercises 159–162, write an algebraic expression that is equivalent to the expression.

159. $\sec[\arcsin(x - 1)]$ **160.** $\tan\left(\arccos \frac{x}{2}\right)$

161. $\sin\left(\arccos \frac{x^2}{4 - x^2}\right)$ **162.** $\csc(\arcsin 10x)$

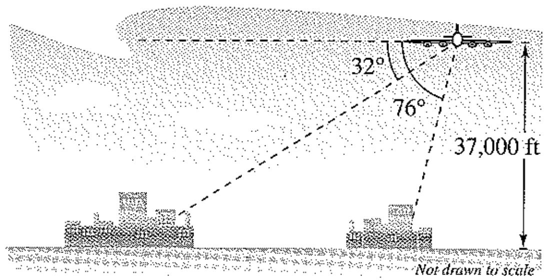
4.8

163. ***Railroad Grade*** A train travels 3.5 kilometers on a straight track with a grade of $1° 10'$. What is the vertical rise of the train in that distance?



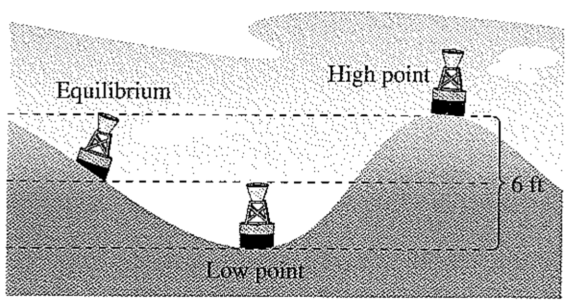
164. ***Mountain Descent*** A road sign at the top of a mountain indicates that for the next 4 miles the grade is 12%. Find the angle of the grade and the change in elevation for a car descending the mountain.

165. ***Distance*** A passenger in an airplane flying at an altitude of 37,000 feet sees two towns directly to the west of the airplane. The angles of depression to the towns are 32° and 76° (see figure). How far apart are the towns?



166. ***Distance*** From city A to city B, a plane flies 650 miles at a bearing of 48°. From city B to city C, the plane flies 810 miles at a bearing of 115°. Find the distance from A to C and the bearing from A to C.

167. ***Wave Motion*** A buoy oscillates in simple harmonic motion as waves go past. At a given time it is noted that the buoy moves a total of 6 feet from its low point to its high point, returning to its high point every 15 seconds (see figure). Write an equation that describes the motion of the buoy if it is at its high point at $t = 0$.



168. ***Wave Motion*** Your fishing bobber oscillates in simple harmonic motion from the waves in the lake where you fish. Your bobber moves a total of 1.5 inches from its high point to its low point and returns to its high point every 3 seconds. Write an equation modeling the motion of your bobber if it is at its high point at $t = 0$.

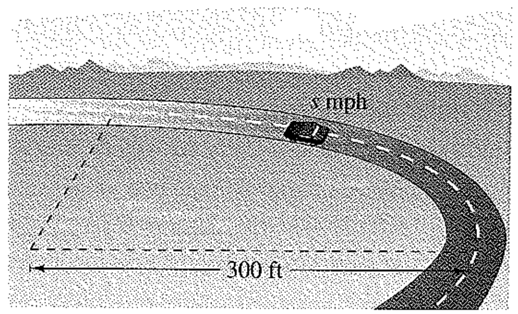
Synthesis

True or False? **In Exercises 169 and 170, determine whether the statement is true or false. Justify your answer.**

169. $y = \sin\theta$ is not a function because $\sin 30° = \sin 150°$.

170. The tangent function is often useful for modeling simple harmonic motion.

171. ***Numerical Analysis*** A 3000-pound automobile is negotiating a circular interchange of radius 300 feet at a speed of s miles per hour (see figure). The relationship between the speed and the angle θ (in degrees) at which the roadway should be banked so that no lateral frictional force is exerted on the tires is $\tan\theta = 0.672s^2/3000$.



(a) Use a graphing utility to complete the table.

s	10	20	30	40	50	60
θ						

(b) In the table, s is incremented by 10, but θ does not increase by equal increments. Explain.

172. ***Approximation*** In calculus it can be shown that the arctangent function can be approximated by the polynomial

$$\arctan x \approx x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7}$$

where x is in radians.

(a) Use a graphing utility to graph the arctangent function and its polynomial approximation in the same viewing window. How do the graphs compare?

(b) Study the pattern in the polynomial approximation of the arctangent function and guess the next term. Then repeat part (a). How does the accuracy of the approximation change when additional terms are added?

4 Chapter Test

Take this test as you would take a test in class. After you are finished, check your work against the answers given in the back of the book.

1. Consider an angle that measures $\frac{5\pi}{4}$ radians.
   (a) Sketch the angle in standard position.
   (b) Determine two coterminal angles (one positive and one negative).
   (c) Convert the angle to degree measure.
2. A truck is moving at a rate of 90 kilometers per hour, and the diameter of its wheels is 1 meter. Find the angular speed of the wheels in radians per minute.
3. Find the exact values of the six trigonometric functions of the angle θ shown in the figure.

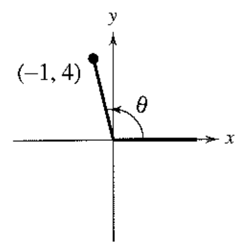


Figure for 3

4. Given that $\tan\theta = \frac{6}{5}$, find the other five trigonometric functions of θ.
5. Determine the reference angle θ' of the angle $\theta = 255°$ and sketch θ and θ' in standard position.
6. Determine the quadrant in which θ lies if $\sec\theta < 0$ and $\tan\theta > 0$.
7. Find two exact values of θ in degrees $(0 \le \theta < 360°)$ if $\cos\theta = -\sqrt{3}/2$.
8. Use a calculator to approximate two values of θ in radians $(0 \le \theta < 2\pi)$ if $\csc\theta = 1.030$. Round your answer to two decimal places.
9. Find the five remaining trigonometric functions of θ, given that $\cos\theta = -\frac{3}{5}$ and $\sin\theta > 0$.

In Exercises 10–15, sketch the graph of the function. (Include two full periods.)

10. $g(x) = -2\sin\left(x - \frac{\pi}{4}\right)$
11. $f(\alpha) = \frac{1}{2}\tan 2\alpha$
12. $f(x) = \frac{1}{2}\sec(x - \pi)$
13. $f(x) = 2\cos(\pi - 2x) + 3$
14. $f(x) = 2\csc\left(x + \frac{\pi}{2}\right)$
15. $f(x) = \frac{1}{4}\cot\left(x - \frac{\pi}{2}\right)$

In Exercises 16 and 17, use a graphing utility to graph the function. If the function is periodic, find its period.

16. $y = \sin 2\pi x + 2\cos \pi x$
17. $y = 6e^{-0.12t}\cos(0.25t),\quad 0 \le t \le 32$
18. Find a, b, and c for the function $f(x) = a\sin(bx + c)$ such that the graph of f matches the graph at the right.

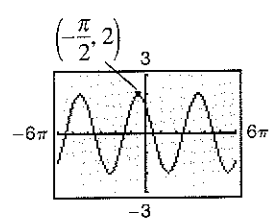


Figure for 18

19. Find the exact value of $\tan\left(\arccos\frac{2}{3}\right)$ without using a calculator.

In Exercises 20–22, use a graphing utility to graph the function.

20. $f(x) = 2\arcsin\left(\frac{1}{2}x\right)$
21. $f(x) = 2\arccos x$
22. $f(x) = \arctan\frac{x}{2}$
23. A plane is 160 miles north and 110 miles east of an airport. What bearing should be taken to fly directly to the airport?