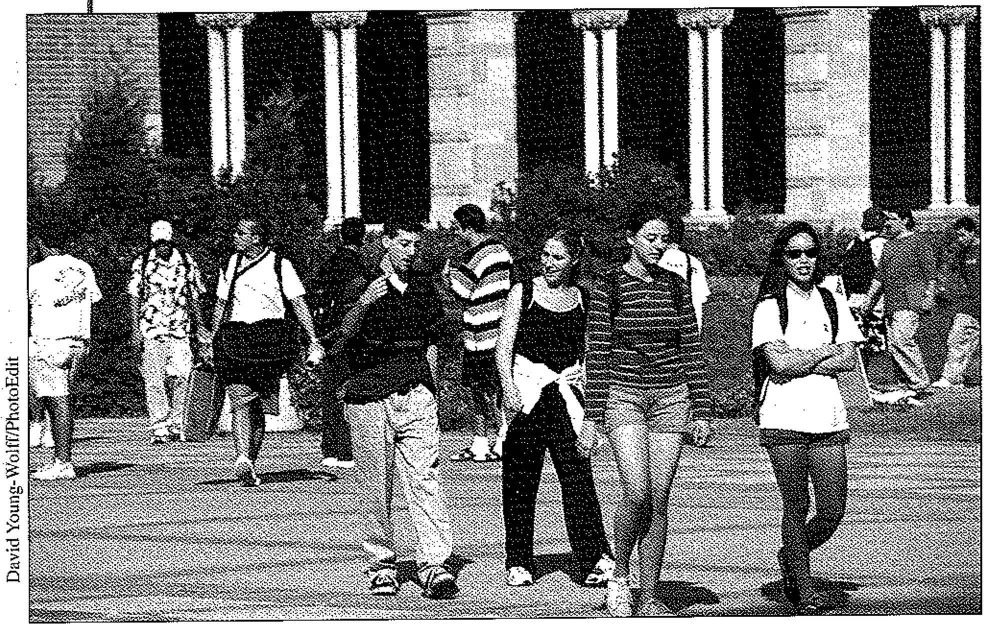
Colleges and universities track enrollment figures in order to determine the financial outlook of the institution. The growth in student enrollment at a college or university can be modeled by a linear equation.


David Young-Wolff/PhotoEdit

1 Functions and Their Graphs

What You Should Learn

In this chapter, you will learn how to:

- Find and use the slope of a line to write and graph linear equations.
- Evaluate functions and find their domains.
- Analyze graphs of functions.
- Identify and graph shifts, reflections, and nonrigid transformations of functions.
- Find arithmetic combinations and compositions of functions.
- Find inverse functions graphically and algebraically.
- Use scatter plots and a graphing utility to find linear models for data.

Introduction to Library of Functions

In Chapter 1, you will be introduced to the concept of a *function*. As you proceed through the text, you will see that functions play a primary role in modeling real-life situations.

There are three basic types of functions that have proven to be the most important in modeling real-life situations. These functions are algebraic functions, exponential and logarithmic functions, and trigonometric and inverse trigonometric functions. These three types of functions are referred to as the *elementary functions*, though they are often placed in the two categories of *algebraic functions* and *transcendental functions*. Each time a new type of function is studied in detail in this text, it will be highlighted in a box similar to this one. The graphs of many of these functions are shown on the inside front cover of this text.

Algebraic Functions

These functions are formed by applying algebraic operations to the identity function $f(x) = x$.

Name	*Function*	*Location*
Linear	$f(x) = ax + b$	Section 1.1
Quadratic	$f(x) = ax^2 + bx + c$	Section 2.1
Cubic	$f(x) = ax^3 + bx^2 + cx + d$	Section 2.2
Polynomial	$P(x) = a_n x^n + a_{n-1}x^{n-1} + \cdot \cdot \cdot + a_2 x^2 + a_1 x + a_0$	Section 2.2
Rational	$f(x) = \dfrac{N(x)}{D(x)}$, $N(x)$ and $D(x)$ are polynomial functions	Section 2.6
Radical	$f(x) = \sqrt[n]{P(x)}$	Section 1.2

Transcendental Functions

These functions cannot be formed from the identity function by using algebraic operations.

Name	*Function*	*Location*
Exponential	$f(x) = a^x, a > 0, a \neq 1$	Section 3.1
Logarithmic	$f(x) = \log_a x, x > 0, a > 0, a \neq 1$	Section 3.2
Trigonometric	$f(x) = \sin x, f(x) = \cos x, f(x) = \tan x,$ $f(x) = \csc x, f(x) = \sec x, f(x) = \cot x$	Section 4.4
Inverse Trigonometric	$f(x) = \arcsin x, f(x) = \arccos x, f(x) = \arctan x$	Section 4.7

Nonelementary Functions

Some useful nonelementary functions include the following.

Name	*Function*	*Location*
Absolute value	$f(x) = \lvert g(x) \rvert$, $g(x)$ is an elementary function	Section 1.2
Piecewise-defined	$f(x) = \begin{cases} 3x + 2, & x \geq 1 \\ -2x + 4, & x < 1 \end{cases}$	Section 1.2
Greatest integer	$f(x) = [\![g(x)]\!]$, $g(x)$ is an elementary function	Section 1.3
Data defined	Formula for temperature: $F = \dfrac{9}{5}C + 32$	Section 1.2

1.1 Lines in the Plane

What you should learn

- Find the slopes of lines.
- Write linear equations given points on lines and their slopes.
- Use slope-intercept forms of linear equations to sketch lines.
- Use slope to identify parallel and perpendicular lines.

Why you should learn it

The slope of a line can be used to solve real-life problems. For instance, Exercise 68 on page 13 shows how to use slope to determine the years in which the earnings per share of stock for Harley-Davidson, Inc. showed the greatest and smallest increase.

Dwayne Newton/PhotoEdit

The Slope of a Line

In this section, you will study lines and their equations. The **slope** of a nonvertical line represents the number of units the line rises or falls vertically for each unit of horizontal change from left to right. For instance, consider the two points (x_1, y_1) and (x_2, y_2) on the line shown in Figure 1.1. As you move from left to right along this line, a change of $(y_2 - y_1)$ units in the vertical direction corresponds to a change of $(x_2 - x_1)$ units in the horizontal direction. That is,

$$y_2 - y_1 = \text{the change in } y$$

and

$$x_2 - x_1 = \text{the change in } x.$$

The slope of the line is given by the ratio of these two changes.

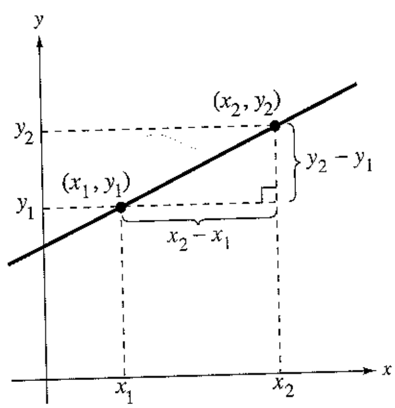


Figure 1.1

Definition of the Slope of a Line

The **slope** m of the nonvertical line through (x_1, y_1) and (x_2, y_2) is

$$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{\text{change in } y}{\text{change in } x}$$

where $x_1 \neq x_2$.

When this formula for slope is used, the *order of subtraction* is important. Given two points on a line, you are free to label either one of them as (x_1, y_1) and the other as (x_2, y_2). However, once you have done this, you must form the numerator and denominator using the same order of subtraction.

$$\underbrace{m = \frac{y_2 - y_1}{x_2 - x_1}}_{\text{Correct}} \qquad \underbrace{m = \frac{y_1 - y_2}{x_1 - x_2}}_{\text{Correct}} \qquad \underbrace{m = \frac{y_2 - y_1}{x_1 - x_2}}_{\text{Incorrect}}$$

Throughout this text, the term *line* always means a *straight* line.

Example 1 Finding the Slope of a Line

Find the slope of the line passing through each pair of points.

a. $(-2, 0)$ and $(3, 1)$ **b.** $(-1, 2)$ and $(2, 2)$ **c.** $(0, 4)$ and $(1, -1)$

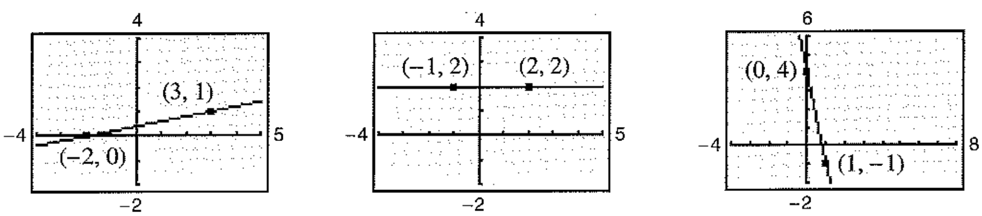
Solution

$$\text{a. } m = \overbrace{\underbrace{\frac{y_2 - y_1}{x_2 - x_1}}_{\text{Difference in } x\text{-values}}}^{\text{Difference in } y\text{-values}} = \frac{1 - 0}{3 - (-2)} = \frac{1}{3 + 2} = \frac{1}{5}$$

$$\text{b. } m = \frac{2 - 2}{2 - (-1)} = \frac{0}{3} = 0$$

$$\text{c. } m = \frac{-1 - 4}{1 - 0} = \frac{-5}{1} = -5$$

The graphs of the three lines are shown in Figure 1.2. Note that the square setting gives the correct "steepness" of the lines.



(a) (b) (c)

Figure 1.2

✔ *Checkpoint* Now try Exercise 9.

> **Exploration**
>
> Use a graphing utility to compare the slopes of the lines $y = 0.5x$, $y = x$, $y = 2x$, and $y = 4x$. What do you observe about these lines? Compare the slopes of the lines $y = -0.5x$, $y = -x$, $y = -2x$, and $y = -4x$. What do you observe about these lines? (*Hint:* Use a square setting to guarantee a true geometric perspective.)

The definition of slope does not apply to vertical lines. For instance, consider the points $(3, 4)$ and $(3, 1)$ on the vertical line shown in Figure 1.3. Applying the formula for slope, you obtain

$$m = \frac{4 - 1}{3 - 3} = \frac{3}{0}. \qquad \text{Undefined}$$

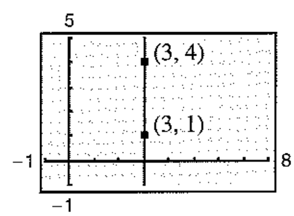


Figure 1.3

Because division by zero is undefined, the slope of a vertical line is undefined.

From the slopes of the lines shown in Figures 1.2 and 1.3, you can make the following generalizations about the slope of a line.

> **The Slope of a Line**
>
> **1.** A line with positive slope $(m > 0)$ *rises* from left to right.
>
> **2.** A line with negative slope $(m < 0)$ *falls* from left to right.
>
> **3.** A line with zero slope $(m = 0)$ is *horizontal.*
>
> **4.** A line with undefined slope is *vertical.*

The Point-Slope Form of the Equation of a Line

If you know the slope of a line *and* you also know the coordinates of one point on the line, you can find an equation for the line. For instance, in Figure 1.4, let (x_1, y_1) be a point on the line whose slope is m. If (x, y) is any *other* point on the line, it follows that

$$\frac{y - y_1}{x - x_1} = m.$$

This equation in the variables x and y can be rewritten in the **point-slope form** of the equation of a line.

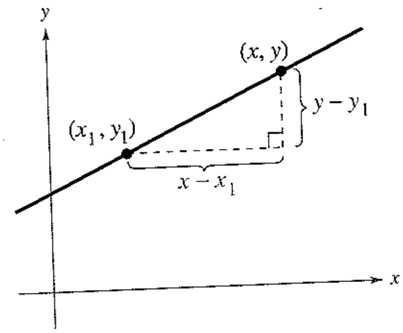


Figure 1.4

> **Point-Slope Form of the Equation of a Line**
>
> The **point-slope form** of the equation of the line that passes through the point (x_1, y_1) and has a slope of m is
>
> $$y - y_1 = m(x - x_1).$$

The point-slope form is most useful for finding the equation of a line if you know at least one point that the line passes through and the slope of the line. You should remember this form of the equation of a line.

Example 2 The Point-Slope Form of the Equation of a Line

Find an equation of the line that passes through the point $(1, -2)$ and has a slope of 3.

Solution

$y - y_1 = m(x - x_1)$	Point-slope form
$y - (-2) = 3(x - 1)$	Substitute for y_1, m, and x_1.
$y + 2 = 3x - 3$	Simplify.
$y = 3x - 5$	Solve for y.

The line is shown in Figure 1.5.

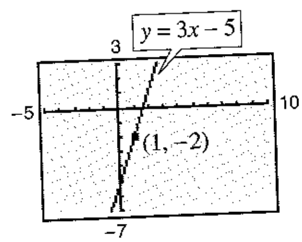


Figure 1.5

✓ *Checkpoint* Now try Exercise 25.

The point-slope form can be used to find an equation of a nonvertical line passing through two points (x_1, y_1) and (x_2, y_2). First, find the slope of the line.

$$m = \frac{y_2 - y_1}{x_2 - x_1}, \; x_1 \neq x_2$$

Then use the point-slope form to obtain the equation

$$y - y_1 = \frac{y_2 - y_1}{x_2 - x_1}(x - x_1).$$

This is sometimes called the **two-point form** of the equation of a line.

STUDY TIP

When you find an equation of the line that passes through two given points, you need to substitute the coordinates of only one of the points into the point-slope form. It does not matter which point you choose because both points will yield the same result.

Example 3 A Linear Model for Sales Prediction

During 2000, Nike's net sales were \$9.0 billion, and in 2001 net sales were \$9.5 billion. Write a linear equation giving the net sales y in terms of the year x. Then use the equation to predict the net sales for 2002. (Source: Nike, Inc.)

Solution

Let $x = 0$ represent 2000. In Figure 1.6, let $(0, 9.0)$ and $(1, 9.5)$ be two points on the line representing the net sales. The slope of this line is

$$m = \frac{9.5 - 9.0}{1 - 0} = 0.5. \qquad m = \frac{y_2 - y_1}{x_2 - x_1}$$

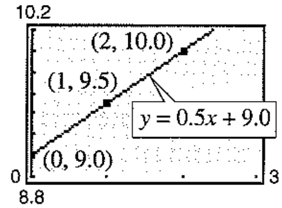


Figure 1.6

By the point-slope form, the equation of the line is as follows.

$$y - 9.0 = 0.5(x - 0) \qquad \text{Write in point-slope form.}$$

$$y = 0.5x + 9.0 \qquad \text{Simplify.}$$

Now, using this equation, you can predict the 2002 net sales ($x = 2$) to be

$$y = 0.5(2) + 9.0 = 1 + 9.0 = \$10.0 \text{ billion.}$$

✓ ***Checkpoint*** Now try Exercise 43.

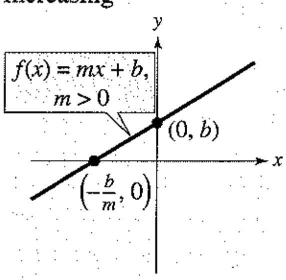
Library of Functions: Linear Function

In the next section, you will be introduced to the precise meaning of the term *function*. The simplest type of function is a *linear function* of the form

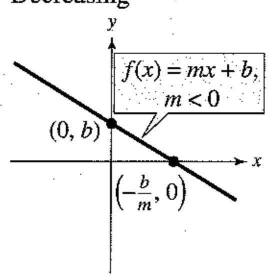
$$f(x) = mx + b.$$

As its name implies, the graph of a linear function is a line that has a slope of m and a y-intercept at $(0, b)$. The basic characteristics of a linear function are summarized below. (Note that some of the terms below will be defined later in the text.)

Graph of $f(x) = mx + b, m > 0$
Domain: $(-\infty, \infty)$
Range: $(-\infty, \infty)$
x-intercept: $(-b/m, 0)$
y-intercept: $(0, b)$
Increasing



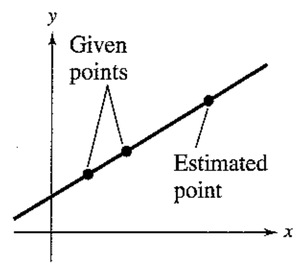
Graph of $f(x) = mx + b, m < 0$
Domain: $(-\infty, \infty)$
Range: $(-\infty, \infty)$
x-intercept: $(-b/m, 0)$
y-intercept: $(0, b)$
Decreasing



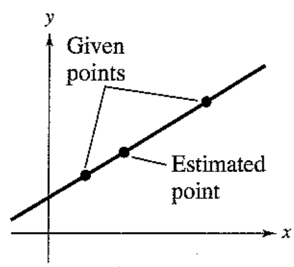
When $m = 0$, the function $f(x) = b$ is called a *constant function* and its graph is a horizontal line.

STUDY TIP

The prediction method illustrated in Example 3 is called **linear extrapolation.** Note in the top figure below that an extrapolated point does not lie between the given points. When the estimated point lies between two given points, as shown in the bottom figure, the procedure used to predict the point is called **linear interpolation.**



Linear Extrapolation



Linear Interpolation

Sketching Graphs of Lines

Many problems in coordinate geometry can be classified as follows.

1. Given a graph (or parts of it), find its equation.
2. Given an equation, sketch its graph.

For lines, the first problem is solved easily by using the point-slope form. This formula, however, is not particularly useful for solving the second type of problem. The form that is better suited to graphing linear equations is the **slope-intercept form** of the equation of a line, $y = mx + b$.

> **Slope-Intercept Form of the Equation of a Line**
>
> The graph of the equation
>
> $y = mx + b$
>
> is a line whose slope is m and whose y-intercept is $(0, b)$.

Example 4 Using the Slope-Intercept Form

Determine the slope and y-intercept of each linear equation. Then describe its graph.

a. $x + y = 2$ **b.** $y = 2$

Algebraic Solution

a. Begin by writing the equation in slope-intercept form.

$x + y = 2$	Write original equation.
$y = 2 - x$	Subtract x from each side.
$y = -x + 2$	Write in slope-intercept form.

From the slope-intercept form of the equation, the slope is -1 and the y-intercept is $(0, 2)$. Because the slope is negative, you know that the graph of the equation is a line that falls one unit for every unit it moves to the right.

b. By writing the equation $y = 2$ in slope-intercept form

$y = (0)x + 2$

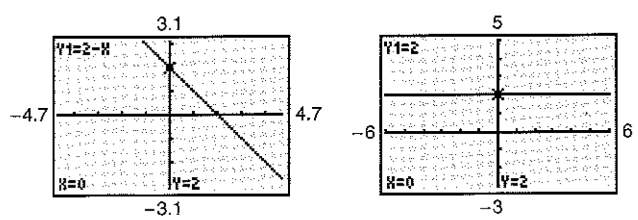
you can see that the slope is 0 and the y-intercept is $(0, 2)$. A zero slope implies that the line is horizontal.

Checkpoint Now try Exercise 45.

Graphical Solution

a. Solve the equation for y to obtain $y = 2 - x$. Enter this equation in your graphing utility. Use a decimal viewing window to graph the equation. To find the y-intercept, use the *value* or *trace* feature. When $x = 0$, $y = 2$, as shown in Figure 1.7(a). So, the y-intercept is $(0, 2)$. To find the slope, continue to use the *trace* feature. Move the cursor along the line until $x = 1$. At this point, $y = 1$. So the graph falls 1 unit for every unit it moves to the right, and the slope is -1.

b. Enter the equation $y = 2$ in your graphing utility and graph the equation. Use the *trace* feature to verify the y-intercept $(0, 2)$ as shown in Figure 1.7(b), and to see that the value of y is the same for all values of x. So, the slope of the horizontal line is 0.



(a) (b)

Figure 1.7

From the slope-intercept form of the equation of a line, you can see that a horizontal line ($m = 0$) has an equation of the form $y = b$. This is consistent with the fact that each point on a horizontal line through $(0, b)$ has a y-coordinate of b. Similarly, each point on a vertical line through $(a, 0)$ has an x-coordinate of a. So, a vertical line has an equation of the form $x = a$. This equation cannot be written in slope-intercept form because the slope of a vertical line is undefined. However, *every* line has an equation that can be written in the **general form**

$$Ax + By + C = 0 \qquad \text{General form of the equation of a line}$$

where A and B are not *both* zero.

Summary of Equations of Lines

1. **General form:** $Ax + By + C = 0$
2. **Vertical line:** $x = a$
3. **Horizontal line:** $y = b$
4. **Slope-intercept form:** $y = mx + b$
5. **Point-slope form:** $y - y_1 = m(x - x_1)$

Exploration

Graph the lines $y_1 = 2x + 1$, $y_2 = \frac{1}{2}x + 1$, and $y_3 = -2x + 1$ in the same viewing window. What do you observe?

Graph the lines $y_1 = 2x + 1$, $y_2 = 2x$, and $y_3 = 2x - 1$ in the same viewing window. What do you observe?

Example 5 Different Viewing Windows

The graphs of the two lines

$$y = -x - 1 \qquad \text{and} \qquad y = -10x - 1$$

are shown in Figure 1.8. Even though the slopes of these lines are quite different (-1 and -10, respectively), the graphs seem misleadingly similar because the viewing windows are different.

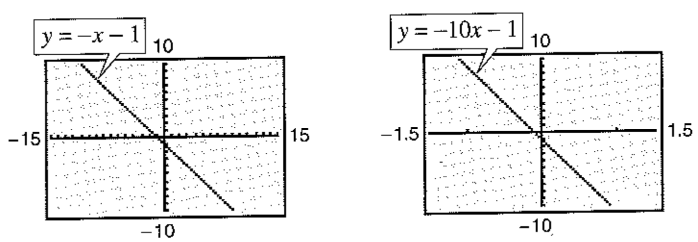


Figure 1.8

✓ *Checkpoint* Now try Exercise 49.

TECHNOLOGY TIP When a graphing utility is used to graph a line, it is important to realize that the graph of the line may not visually appear to have the slope indicated by its equation. This occurs because of the viewing window used for the graph. For instance, Figure 1.9 shows graphs of $y = 2x + 1$ produced on a graphing utility using three different viewing windows. Notice that the slopes in Figures 1.9(a) and (b) do not visually appear to be equal to 2. However, if you use a *square setting*, as in Figure 1.9(c), the slope visually appears to be 2.

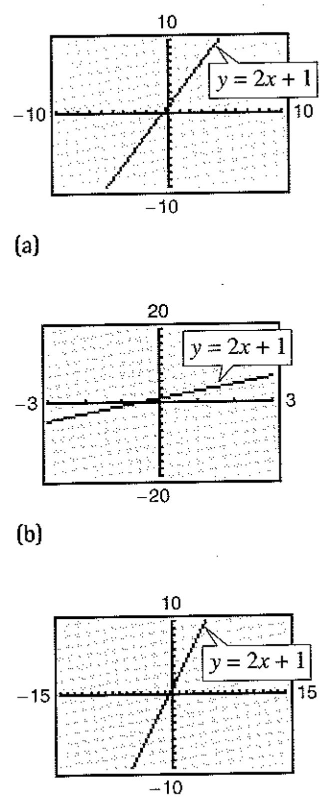


Figure 1.9

Parallel and Perpendicular Lines

The slope of a line is a convenient tool for determining whether two lines are parallel or perpendicular.

> **Parallel Lines**
>
> Two distinct nonvertical lines are **parallel** if and only if their slopes are equal. That is,
>
> $m_1 = m_2.$

Example 6 Equations of Parallel Lines

Find the slope-intercept form of the equation of the line that passes through the point $(2, -1)$ and is parallel to the line $2x - 3y = 5$.

Solution

Begin by writing the equation of the given line in slope-intercept form.

$2x - 3y = 5$ Write original equation.

$-2x + 3y = -5$ Multiply by -1.

$3y = 2x - 5$ Add $2x$ to each side.

$y = \frac{2}{3}x - \frac{5}{3}$ Write in slope-intercept form.

Therefore, the given line has a slope of $m = \frac{2}{3}$. Any line parallel to the given line must also have a slope of $\frac{2}{3}$. So, the line through $(2, -1)$ has the following equation.

$y - (-1) = \frac{2}{3}(x - 2)$ Write in point-slope form.

$y + 1 = \frac{2}{3}x - \frac{4}{3}$ Simplify.

$y = \frac{2}{3}x - \frac{7}{3}$ Write in slope-intercept form.

Notice the similarity between the slope-intercept form of the original equation and the slope-intercept form of the parallel equation. The graphs of both equations are shown in Figure 1.10.

✓ ***Checkpoint*** Now try Exercise 55(a).

> **Perpendicular Lines**
>
> Two nonvertical lines are **perpendicular** if and only if their slopes are negative reciprocals of each other. That is,
>
> $m_1 = -\frac{1}{m_2}.$

TECHNOLOGY TIP

Be careful when you graph equations such as $y = \frac{2}{3}x - \frac{7}{3}$ on your graphing utility. A common mistake is to type in the equation as

Y1 = 2/3X − 7/3,

which may not be interpreted by your graphing utility as the original equation. You should use one of the following formulas.

Y1 = 2X/3 − 7/3

Y1 = (2/3)X − 7/3

Do you see why?

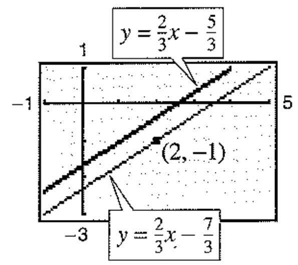


Figure 1.10

Example 7 Equations of Perpendicular Lines

Find the slope-intercept form of the equation of the line that passes through the point $(2, -1)$ and is perpendicular to the line $2x - 3y = 5$.

Solution

From Example 6, you know that the equation can be written in the slope-intercept form $y = \frac{2}{3}x - \frac{5}{3}$. You can see that the line has a slope of $\frac{2}{3}$. So, any line perpendicular to this line must have a slope of $-\frac{3}{2}$ (because $-\frac{3}{2}$ is the negative reciprocal of $\frac{2}{3}$). So, the line through the point $(2, -1)$ has the following equation.

$$y - (-1) = -\tfrac{3}{2}(x - 2) \qquad \text{Write in point-slope form.}$$

$$y + 1 = -\tfrac{3}{2}x + 3 \qquad \text{Simplify.}$$

$$y = -\tfrac{3}{2}x + 2 \qquad \text{Write in slope-intercept form.}$$

The graphs of both equations are shown in Figure 1.11.

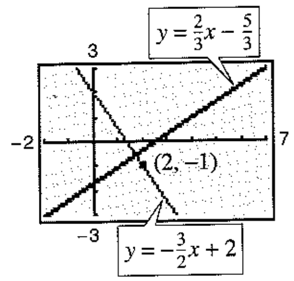


Figure 1.11

✓ *Checkpoint* Now try Exercise 55(b).

Example 8 Graphs of Perpendicular Lines

Use a graphing utility to graph the lines

$$y = x + 1$$

and

$$y = -x + 3$$

in the same viewing window. The lines are supposed to be perpendicular (they have slopes of $m_1 = 1$ and $m_2 = -1$). Do they appear to be perpendicular on the display?

Solution

If the viewing window is nonsquare, as in Figure 1.12, the two lines will not appear perpendicular. If, however, the viewing window is square, as in Figure 1.13, the lines will appear perpendicular.

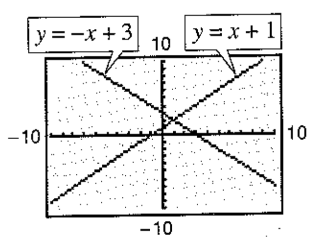


Figure 1.12

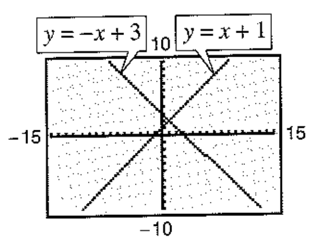


Figure 1.13

✓ *Checkpoint* Now try Exercise 61.

1.1 Exercises

Vocabulary Check

1. Match each equation with its form.

(a) $Ax + By + C = 0$ — (i) vertical line

(b) $x = a$ — (ii) slope-intercept form

(c) $y = b$ — (iii) general form

(d) $y = mx + b$ — (iv) point-slope form

(e) $y - y_1 = m(x - x_1)$ — (v) horizontal line

In Exercises 2–5, fill in the blanks.

2. For a line, the ratio of the change in y to the change in x is called the _______ of the line.

3. Two lines are _______ if and only if their slopes are equal.

4. Two lines are _______ if and only if their slopes are negative reciprocals of each other.

5. The prediction method _______ is the method used to estimate a point on a line that does not lie between the given points.

In Exercises 1 and 2, identify the line that has the indicated slope.

1. (a) $m = \frac{2}{3}$ (b) m is undefined. (c) $m = -2$

2. (a) $m = 0$ (b) $m = -\frac{3}{4}$ (c) $m = 1$

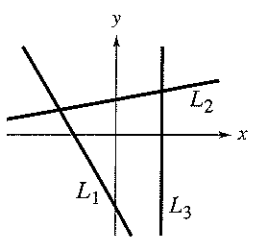


Figure for 1

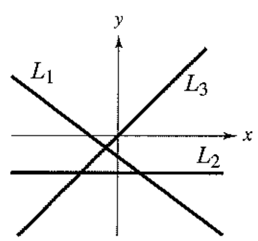


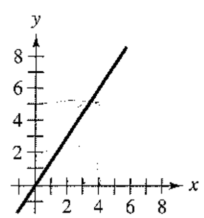
Figure for 2

In Exercises 3 and 4, sketch the lines through the point with the indicated slopes on the same set of coordinate axes.

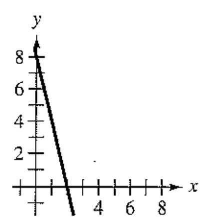
Point	*Slopes*	
3. (2, 3)	(a) 0	(b) 1
	(c) 2	(d) −3
4. (−4, 1)	(a) 3	(b) −3
	(c) $\frac{1}{2}$	(d) Undefined

In Exercises 5 and 6, estimate the slope of the line.

5.



6.



In Exercises 7–10, use a graphing utility to plot the points and use the *draw* feature to graph the line segment connecting the two points. (Use a square setting.) Then find the slope of the line passing through the pair of points.

7. (0, −10), (−4, 0) **8.** (2, 4), (4, −4)

9. (−6, −1), (−6, 4) **10.** (−3, −2), (1, 6)

In Exercises 11–18, use the point on the line and the slope of the line to find three additional points through which the line passes. (There are many correct answers.)

Point	*Slope*
11. (2, 1)	$m = 0$
12. (3, −2)	$m = 0$
13. (1, 5)	m is undefined.
14. (−4, 1)	m is undefined.

	Point	Slope
15.	$(0, -9)$	$m = -2$
16.	$(-5, 4)$	$m = 2$
17.	$(7, -2)$	$m = \frac{1}{2}$
18.	$(-1, -6)$	$m = -\frac{1}{2}$

In Exercises 19–24, (a) find the slope and y-intercept (if possible) of the equation of the line algebraically, (b) sketch the line by hand, and (c) use a graphing utility to verify your answers to parts (a) and (b).

19. $5x - y + 3 = 0$ **20.** $2x + 3y - 9 = 0$

21. $5x - 2 = 0$ **22.** $3x + 7 = 0$

23. $3y + 5 = 0$ **24.** $-11 - 8y = 0$

In Exercises 25–32, find the general form of the equation of the line that passes through the given point and has the indicated slope. Sketch the line by hand. Use a graphing utility to verify your sketch, if possible.

	Point	Slope
25.	$(0, -2)$	$m = 3$
26.	$(-3, 6)$	$m = -2$
27.	$(0, 0)$	$m = 4$
28.	$(-2, -5)$	$m = \frac{3}{4}$
29.	$(6, -1)$	m is undefined.
30.	$(-10, 4)$	m is undefined.
31.	$\left(-\frac{1}{2}, \frac{3}{2}\right)$	$m = 0$
32.	$(2.3, -8.5)$	$m = 0$

In Exercises 33–42, find the slope-intercept form of the equation of the line that passes through the points. Use a graphing utility to graph the line.

33. $(5, -1), (-5, 5)$ **34.** $(4, 3), (-4, -4)$

35. $(-8, 1), (-8, 7)$ **36.** $(-1, 4), (6, 4)$

37. $\left(2, \frac{1}{2}\right), \left(\frac{1}{2}, \frac{5}{4}\right)$ **38.** $(1, 1), \left(6, -\frac{2}{3}\right)$

39. $\left(-\frac{1}{10}, -\frac{3}{5}\right), \left(\frac{9}{10}, -\frac{9}{5}\right)$ **40.** $\left(\frac{3}{4}, \frac{3}{2}\right), \left(-\frac{4}{3}, \frac{7}{4}\right)$

41. $(1, 0.6), (-2, -0.6)$ **42.** $(-8, 0.6), (2, -2.4)$

43. ***Annual Salary*** A jeweler's salary was \$28,500 in 2000 and \$32,900 in 2002. The jeweler's salary follows a linear growth pattern. What will the jeweler's salary be in 2006?

44. ***Annual Salary*** A librarian's salary was \$25,000 in 2000 and \$27,500 in 2002. The librarian's salary follows a linear growth pattern. What will the librarian's salary be in 2006?

In Exercises 45–48, determine the slope and y-intercept of the linear equation. Then describe its graph.

45. $x - 2y = 4$ **46.** $3x + 4y = 1$

47. $x = -6$ **48.** $y = 12$

In Exercises 49 and 50, use a graphing utility to graph the equation using each of the suggested viewing windows. Describe the difference between the two graphs.

49. $y = 0.5x - 3$

Xmin = -5 Xmax = 10 Xscl = 1 Ymin = -1 Ymax = 10 Yscl = 1	Xmin = -2 Xmax = 10 Xscl = 1 Ymin = -4 Ymax = 1 Yscl = 1

50. $y = -8x + 5$

Xmin = -5 Xmax = 5 Xscl = 1 Ymin = -10 Ymax = 10 Yscl = 1	Xmin = -5 Xmax = 10 Xscl = 1 Ymin = -80 Ymax = 80 Yscl = 20

In Exercises 51–54, determine whether the lines L_1 and L_2 passing through the pairs of points are parallel, perpendicular, or neither.

51. L_1: $(0, -1), (5, 9)$
L_2: $(0, 3), (4, 1)$

52. L_1: $(-2, -1), (1, 5)$
L_2: $(1, 3), (5, -5)$

53. L_1: $(3, 6), (-6, 0)$
L_2: $(0, -1), \left(5, \frac{7}{3}\right)$

54. L_1: $(4, 8), (-4, 2)$
L_2: $(3, -5), \left(-1, \frac{1}{3}\right)$

In Exercises 55–60, write the slope-intercept forms of the equations of the lines through the given point (a) parallel to the given line and (b) perpendicular to the given line.

	Point	Line
55.	$(2, 1)$	$4x - 2y = 3$
56.	$(-3, 2)$	$x + y = 7$
57.	$\left(-\frac{2}{3}, \frac{7}{8}\right)$	$3x + 4y = 7$
58.	$(-3.9, -1.4)$	$6x + 2y = 9$
59.	$(3, -2)$	$x - 4 = 0$
60.	$(-4, 1)$	$y + 2 = 0$

Graphical Analysis **In Exercises 61–64, identify any relationships that exist among the lines, and then use a graphing utility to graph the three equations in the same viewing window. Adjust the viewing window so that each slope appears visually correct. Use the slopes of the lines to verify your results.**

61. (a) $y = 2x$ (b) $y = -2x$ (c) $y = \frac{1}{2}x$

62. (a) $y = \frac{2}{3}x$ (b) $y = -\frac{3}{2}x$ (c) $y = \frac{2}{3}x + 2$

63. (a) $y = -\frac{1}{2}x$ (b) $y = -\frac{1}{2}x + 3$ (c) $y = 2x - 4$

64. (a) $y = x - 8$ (b) $y = x + 1$ (c) $y = -x + 3$

65. ***Sales*** The following are the slopes of lines representing annual sales y in terms of time x in years. Use each slope to interpret any change in annual sales for a one-year increase in time.

(a) The line has a slope of $m = 135$.

(b) The line has a slope of $m = 0$.

(c) The line has a slope of $m = -40$.

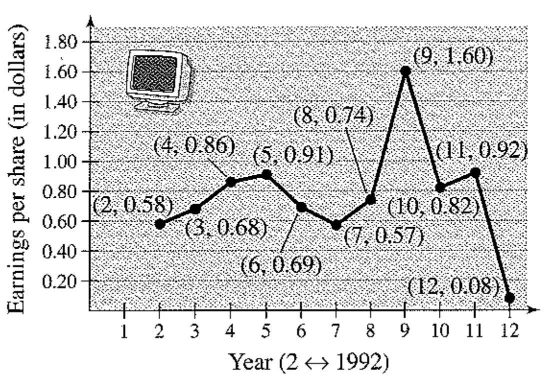
66. ***Revenue*** The following are the slopes of lines representing daily revenues y in terms of time x in days. Use each slope to interpret any change in daily revenues for a one-day increase in time.

(a) The line has a slope of $m = 400$.

(b) The line has a slope of $m = 100$.

(c) The line has a slope of $m = 0$.

67. ***Earnings per Share*** The graph shows the earnings per share of stock for Circuit City for the years 1992 through 2002. (Source: Circuit City Stores, Inc.)



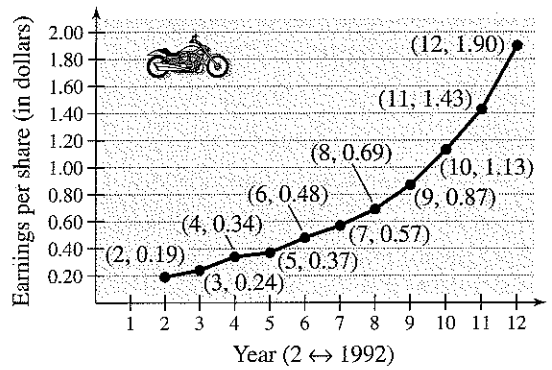
(a) Use the slopes to determine the year(s) in which the earnings per share of stock showed the greatest increase and decrease.

(b) Find the equation of the line between the years 1992 and 2002.

(c) Interpret the meaning of the slope of the equation from part (b) in the context of the problem.

(d) Use the equation from part (b) to estimate the earnings per share of stock for the year 2006. Do you think this is an accurate estimation? Explain.

68. ***Earnings per Share*** The graph shows the earnings per share of stock for Harley-Davidson, Inc. for the years 1992 through 2002. (Source: Harley-Davidson, Inc.)



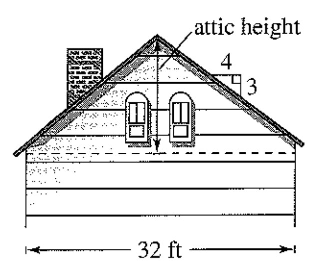
(a) Use the slopes to determine the years in which the earnings per share of stock showed the greatest increase and the smallest increase.

(b) Find the equation of the line between the years 1992 and 2002.

(c) Interpret the meaning of the slope of the equation from part (b) in the context of the problem.

(d) Use the equation from part (b) to estimate the earnings per share of stock for the year 2006. Do you think this is an accurate estimation? Explain.

69. ***Height*** The "rise to run" ratio of the roof of a house determines the steepness of the roof. The rise to run ratio of a roof is 3 to 4. Determine the maximum height in the attic of the house if the house is 32 feet wide.

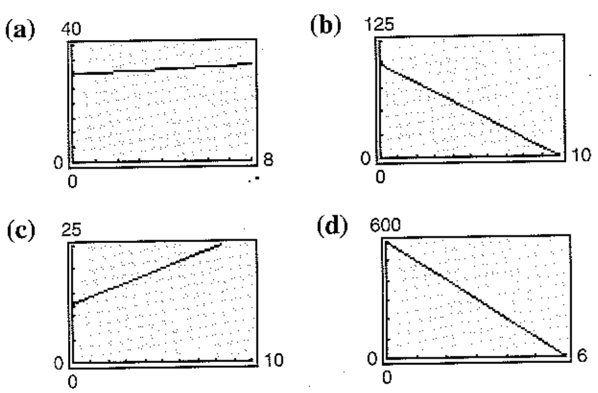


70. ***Road Grade*** When driving down a mountain road, you notice warning signs indicating that it is a "12% grade." This means that the slope of the road is $-\frac{12}{100}$. Approximate the amount of horizontal change in your position if you note from elevation markers that you have descended 2000 feet vertically.

Rate of Change **In Exercises 71–74, you are given the dollar value of a product in 2004 and the rate at which the value of the product is expected to change during the next 5 years. Write a linear equation that gives the dollar value V of the product in terms of the year t. (Let $t = 4$ represent 2004.)**

	2004 Value	*Rate*
71.	\$2540	\$125 increase per year
72.	\$156	\$4.50 increase per year
73.	\$20,400	\$2000 decrease per year
74.	\$245,000	\$5600 decrease per year

Graphical Interpretation **In Exercises 75–78, match the description with its graph. Determine the slope of each graph and how it is interpreted in the given context. [The graphs are labeled (a), (b), (c), and (d).]**



75. You are paying \$10 per week to repay a \$100 loan.

76. An employee is paid \$12.50 per hour plus \$1.50 for each unit produced per hour.

77. A sales representative receives \$30 per day for food plus \$0.35 for each mile traveled.

78. A word processor that was purchased for \$600 depreciates \$100 per year.

79. ***Meteorology*** Find the equation of the line that shows the relationship between the temperature in degrees Celsius C and degrees Fahrenheit F. Remember that water freezes at 0°C (32°F) and boils at 100°C (212°F).

80. ***Meteorology*** Use the result of Exercise 79 to complete the table.

C		−10°	10°			177°
F	0°			68°	90°	

81. ***Depreciation*** A pizza shop purchases a used pizza oven for \$875. After 5 years, the oven will have to be replaced.

(a) Write a linear equation giving the value V of the oven during the 5 years it will be used.

(b) Use a graphing utility to graph the linear equation representing the depreciation of the oven, and use the *value* or *trace* feature to complete the table.

t	0	1	2	3	4	5
V						

(c) Verify your answers in part (b) algebraically by using the equation you found in part (a).

82. ***Depreciation*** A school district purchases a high-volume printer, copier, and scanner for \$25,000. After 10 years, the equipment will have to be replaced. Its value at that time is expected to be \$2000.

(a) Write a linear equation giving the value V of the equipment during the 10 years it will be used.

(b) Use a graphing utility to graph the linear equation representing the depreciation of the equipment, and use the *value* or *trace* feature to complete the table.

t	0	1	2	3	4	5	6	7	8	9	10
V											

(c) Verify your answers in part (b) algebraically by using the equation you found in part (a).

83. ***Cost, Revenue, and Profit*** A contractor purchases a bulldozer for \$36,500. The bulldozer requires an average expenditure of \$5.25 per hour for fuel and maintenance, and the operator is paid \$11.50 per hour.

(a) Write a linear equation giving the total cost C of operating the bulldozer for t hours. (Include the purchase cost of the bulldozer.)

(b) Assuming that customers are charged \$27 per hour of bulldozer use, write an equation for the revenue R derived from t hours of use.

(c) Use the profit formula $(P = R - C)$ to write an equation for the profit derived from t hours of use.

(d) Use the result of part (c) to find the break-even point (the number of hours the bulldozer must be used to yield a profit of 0 dollars).

84. ***Rental Demand*** A real estate office handles an apartment complex with 50 units. When the rent per unit is \$580 per month, all 50 units are occupied. However, when the rent is \$625 per month, the average number of occupied units drops to 47. Assume that the relationship between the monthly rent p and the demand x is linear.

(a) Write the equation of the line giving the demand x in terms of the rent p.

(b) Use a graphing utility to graph the demand equation and use the *trace* feature to estimate the number of units occupied when the rent is \$655. Verify your answer algebraically.

(c) Use the demand equation to predict the number of units occupied when the rent is lowered to \$595. Verify your answer graphically.

85. ***Education*** In 1990, Penn State University had an enrollment of 75,365 students. By 2002, the enrollment had increased to 83,038. (Source: Penn State Fact Book)

(a) What was the average annual change in enrollment from 1990 to 2002?

(b) Use the average annual change in enrollment to estimate the enrollments in 1984, 1997, and 2000.

(c) Write the equation of a line that represents the given data. What is its slope? Interpret the slope in the context of the problem.

86. ***Writing*** Using the results from Exercise 85, write a short paragraph discussing the concepts of *slope* and *average rate of change*.

Synthesis

True or False? **In Exercises 87 and 88, determine whether the statement is true or false. Justify your answer.**

87. The line through $(-8, 2)$ and $(-1, 4)$ and the line through $(0, -4)$ and $(-7, 7)$ are parallel.

88. If the points $(10, -3)$ and $(2, -9)$ lie on the same line, then the point $\left(-12, -\frac{37}{2}\right)$ also lies on that line.

Exploration **In Exercises 89 and 90, use the values of a and b and a graphing utility to graph the equation of the line**

$$\frac{x}{a} + \frac{y}{b} = 1, \qquad a \neq 0, b \neq 0.$$

Use the graphs to make a conjecture about what a and b represent. Verify your conjecture.

89. $a = 5, \quad b = -3$

90. $a = -6, \quad b = 2$

In Exercises 91–94, use the results of Exercises 89 and 90 to write an equation of the line that passes through the points.

91. x-intercept: $(2, 0)$
y-intercept: $(0, 3)$

92. x-intercept: $(-5, 0)$
y-intercept: $(0, -4)$

93. x-intercept: $\left(-\frac{1}{6}, 0\right)$
y-intercept: $\left(0, -\frac{2}{3}\right)$

94. x-intercept: $\left(\frac{3}{4}, 0\right)$
y-intercept: $\left(0, \frac{4}{5}\right)$

95. ***Think About It*** The slopes of two lines are -3 and $\frac{5}{2}$. Which is steeper?

96. ***Think About It*** Is it possible for two lines with positive slopes to be perpendicular? Explain.

97. ***Writing*** Explain how you could show that the points $A(2, 3)$, $B(2, 9)$, and $C(7, 3)$ are the vertices of a right triangle.

98. ***Writing*** Write a brief paragraph explaining whether or not any pair of points on a line can be used to calculate the slope of the line.

Review

In Exercises 99–104, determine whether the expression is a polynomial. If it is, write the polynomial in standard form.

99. $x + 20$

100. $3x - 10x^2 + 1$

101. $4x^2 + x^{-1} - 3$

102. $2x^2 - 2x^4 - x^3 + 2$

103. $\dfrac{x^2 + 3x + 4}{x^2 - 9}$

104. $\sqrt{x^2 + 7x + 6}$

In Exercises 105–108, factor the trinomial.

105. $x^2 - 6x - 27$

106. $x^2 - 11x + 28$

107. $2x^2 + 11x - 40$

108. $3x^2 - 16x + 5$

1.2 Functions

Introduction to Functions

Many everyday phenomena involve pairs of quantities that are related to each other by some rule of correspondence. The mathematical term for such a rule of correspondence is a **relation.** Here are two examples.

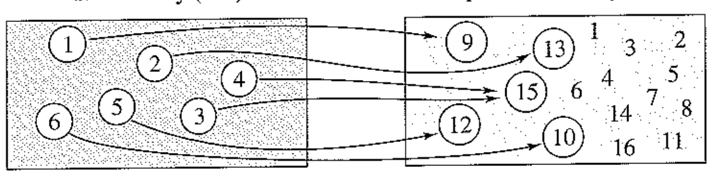
1. The simple interest I earned on an investment of \$1000 for 1 year is related to the annual interest rate r by the formula $I = 1000r$.
2. The area A of a circle is related to its radius r by the formula $A = \pi r^2$.

Not all relations have simple mathematical formulas. For instance, people commonly match up NFL starting quarterbacks with touchdown passes, and hours of the day with temperature. In each of these cases, there is some relation that matches each item from one set with exactly one item from a different set. Such a relation is called a **function.**

> **Definition of a Function**
>
> A **function** f from a set A to a set B is a relation that assigns to each element x in the set A exactly one element y in the set B. The set A is the **domain** (or set of inputs) of the function f, and the set B contains the **range** (or set of outputs).

To help understand this definition, look at the function that relates the time of day to the temperature in Figure 1.14.

Time of day (P.M.) Temperature (in degrees C)



Set A is the domain.
Inputs: 1, 2, 3, 4, 5, 6

Set B contains the range.
Outputs: 9, 10, 12, 13, 15

Figure 1.14

This function can be represented by the ordered pairs {(1, 9°), (2, 13°), (3, 15°), (4, 15°), (5, 12°), (6, 10°)}. In each ordered pair, the first coordinate (x-value) is the **input** and the second coordinate (y-value) is the **output.**

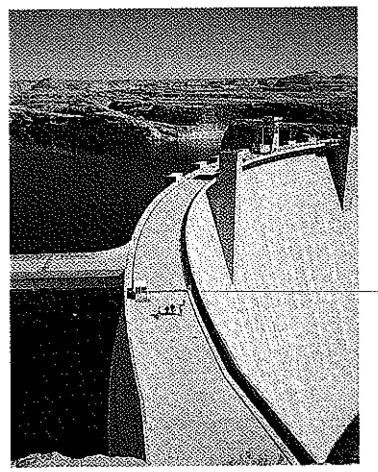
> **Characteristics of a Function from Set A to Set B**
>
> 1. Each element of A must be matched with an element of B.
> 2. Some elements of B may not be matched with any element of A.
> 3. Two or more elements of A may be matched with the same element of B.
> 4. An element of A (the domain) cannot be matched with two different elements of B.

What you should learn

- Decide whether relations between two variables represent a function.
- Use function notation and evaluate functions.
- Find the domains of functions.
- Use functions to model and solve real-life problems.
- Evaluate difference quotients.

Why you should learn it

Many natural phenomena can be modeled by functions, such as the force of water against the face of a dam, explored in Exercise 81 on page 28.



Kunio Owaki/Corbis

Library of Functions: Data Defined Function

Many functions do not have simple mathematical formulas, but are defined by real-life data. Such functions arise when you are using collections of data to model real-life applications. Functions can be represented in four ways.

1. *Verbally* by a sentence that describes how the input variable is related to the output variable

   Example: The input value x is the election year from 1952 to 2004 and the output value y is the elected president of the United States.
2. *Numerically* by a table or a list of ordered pairs that matches input values with output values

   Example: In the set of ordered pairs $\{(2, 34), (4, 40), (6, 45), (8, 50), (10, 54)\}$, the input value is the age of a male child in years and the output value is the height of the child in inches.
3. *Graphically* by points on a graph in a coordinate plane in which the input values are represented by the horizontal axis and the output values are represented by the vertical axis

   Example: See Figure 1.15.
4. *Algebraically* by an equation in two variables

   Example: The formula for temperature, $F = \frac{9}{5}C + 32$, where F is the temperature in degrees Fahrenheit and C is the temperature in degrees Celsius, is an equation that represents a function. You will see that it is often convenient to approximate data using a mathematical model or formula.

STUDY TIP

To determine whether or not a relation is a function, you must decide whether each input value is matched with exactly one output value. If any input value is matched with two or more output values, the relation is not a function.

Example 1 Testing for Functions

Decide whether the relation represents y as a function of x.

a.

Input x	2	2	3	4	5
Output y	11	10	8	5	1

b.

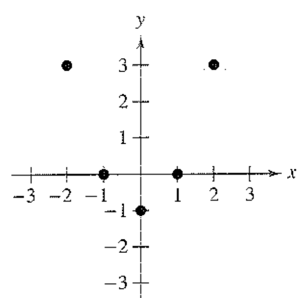


Figure 1.15

Solution

a. This table *does not* describe y as a function of x. The input value 2 is matched with two different y-values.

b. The graph in Figure 1.15 *does* describe y as a function of x. Each input value is matched with exactly one output value.

✓ ***Checkpoint*** Now try Exercise 5.

STUDY TIP

Be sure you see that the *range* of a function is not the same as the use of *range* relating to the viewing window of a graphing utility.

In algebra, it is common to represent functions by equations or formulas involving two variables. For instance, the equation $y = x^2$ represents the variable y as a function of the variable x. In this equation, x is the **independent variable** and y is the **dependent variable.** The domain of the function is the set of all values taken on by the independent variable x, and the range of the function is the set of all values taken on by the dependent variable y.

Example 2 Testing for Functions Represented Algebraically

Which of the equations represent(s) y as a function of x?

a. $x^2 + y = 1$ **b.** $-x + y^2 = 1$

Solution

To determine whether y is a function of x, try to solve for y in terms of x.

a. Solving for y yields

$$x^2 + y = 1 \qquad \text{Write original equation.}$$

$$y = 1 - x^2. \qquad \text{Solve for } y.$$

Each value of x corresponds to exactly one value of y. So, y is a function of x.

b. Solving for y yields

$$-x + y^2 = 1 \qquad \text{Write original equation.}$$

$$y^2 = 1 + x \qquad \text{Add } x \text{ to each side.}$$

$$y = \pm\sqrt{1 + x}. \qquad \text{Solve for } y.$$

The $\pm$ indicates that for a given value of x there correspond two values of y. For instance, when $x = 3$, $y = 2$ or $y = -2$. So, y is not a function of x.

☑ *Checkpoint* Now try Exercise 19.

Exploration

Use a graphing utility to graph $x^2 + y = 1$. Then use the graph to write a convincing argument that each x-value has at most one y-value.

Use a graphing utility to graph $-x + y^2 = 1$. (*Hint:* You will need to use two equations.) Does the graph represent y as a function of x? Explain.

Function Notation

When an equation is used to represent a function, it is convenient to name the function so that it can be referenced easily. For example, you know that the equation $y = 1 - x^2$ describes y as a function of x. Suppose you give this function the name "f." Then you can use the following **function notation.**

Input	*Output*	*Equation*
x	$f(x)$	$f(x) = 1 - x^2$

The symbol $f(x)$ is read as the *value of f at x* or simply *f of x*. The symbol $f(x)$ corresponds to the y-value for a given x. So, you can write $y = f(x)$. Keep in mind that f is the *name* of the function, whereas $f(x)$ is the *output value* of the function at the *input value* x. In function notation, the *input* is the independent variable and the *output* is the dependent variable. For instance, the function $f(x) = 3 - 2x$ has *function values* denoted by $f(-1)$, $f(0)$, and so on. To find these values, substitute the specified input values into the given equation.

For $x = -1$, $\quad f(-1) = 3 - 2(-1) = 3 + 2 = 5.$

For $x = 0$, $\quad f(0) = 3 - 2(0) = 3 - 0 = 3.$

TECHNOLOGY TIP

You can use a graphing utility to evaluate a function. Use the Evaluating an Algebraic Expression Program found on the website *college.hmco.com.* The program will prompt you for a value of x, and then evaluate the expression in the equation editor for that value of x. Try using the program to evaluate several different functions of x.

Although f is often used as a convenient function name and x is often used as the independent variable, you can use other letters. For instance,

$$f(x) = x^2 - 4x + 7, \quad f(t) = t^2 - 4t + 7, \quad \text{and} \quad g(s) = s^2 - 4s + 7$$

all define the same function. In fact, the role of the independent variable is that of a "placeholder." Consequently, the function could be described by

$$f(\quad) = (\quad)^2 - 4(\quad) + 7.$$

Example 3 Evaluating a Function

Let $g(x) = -x^2 + 4x + 1$. Find (a) $g(2)$, (b) $g(t)$, and (c) $g(x + 2)$.

Solution

a. Replacing x with 2 in $g(x) = -x^2 + 4x + 1$ yields the following.

$$g(2) = -(2)^2 + 4(2) + 1 = -4 + 8 + 1 = 5$$

b. Replacing x with t yields the following.

$$g(t) = -(t)^2 + 4(t) + 1 = -t^2 + 4t + 1$$

c. Replacing x with $x + 2$ yields the following.

$$g(x + 2) = -(x + 2)^2 + 4(x + 2) + 1 \qquad \text{Substitute } x + 2 \text{ for } x.$$
$$= -(x^2 + 4x + 4) + 4x + 8 + 1 \qquad \text{Multiply.}$$
$$= -x^2 - 4x - 4 + 4x + 8 + 1 \qquad \text{Distributive Property}$$
$$= -x^2 + 5 \qquad \text{Simplify.}$$

✓ *Checkpoint* Now try Exercise 33.

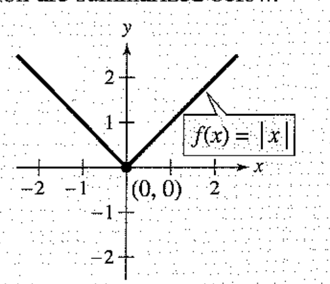
In Example 3, note that $g(x + 2)$ is not equal to $g(x) + g(2)$. In general, $g(u + v) \neq g(u) + g(v)$.

Library of Functions: Piecewise-Defined Function

A *piecewise-defined function* is a function that is defined by two or more equations over a specified domain. The *absolute value function* given by $f(x) = |x|$ can be written as a piecewise-defined function. The basic characteristics of the absolute value function are summarized below.

Graph of $f(x) = |x| = \begin{cases} x, & x \geq 0 \\ -x, & x < 0 \end{cases}$

Domain: $(-\infty, \infty)$
Range: $[0, \infty)$
Intercept: $(0, 0)$
Decreasing on $(-\infty, 0)$
Increasing on $(0, \infty)$



An illustration of a piecewise-defined function is given in Example 4.

Example 4 A Piecewise-Defined Function

Evaluate the function when $x = -1$ and 0.

$$f(x) = \begin{cases} x^2 + 1, & x < 0 \\ x - 1, & x \geq 0 \end{cases}$$

Solution

Because $x = -1$ is less than 0, use $f(x) = x^2 + 1$ to obtain

$$f(-1) = (-1)^2 + 1 = 2.$$

For $x = 0$, use $f(x) = x - 1$ to obtain

$$f(0) = (0) - 1 = -1.$$

✓ *Checkpoint* Now try Exercise 37.

TECHNOLOGY TIP

Most graphing utilities can graph piecewise-defined functions. For instructions on how to enter a piecewise-defined function into your graphing utility, consult your user's manual. You may find it helpful to set your graphing utility to *dot mode* before graphing.

The Domain of a Function

The domain of a function can be described explicitly or it can be *implied* by the expression used to define the function. The **implied domain** is the set of all real numbers for which the expression is defined. For instance, the function

$$f(x) = \frac{1}{x^2 - 4}$$ Domain excludes x-values that result in division by zero.

has an implied domain that consists of all real x other than $x = \pm 2$. These two values are excluded from the domain because division by zero is undefined. Another common type of implied domain is that used to avoid even roots of negative numbers. For example, the function

$$f(x) = \sqrt{x}$$ Domain excludes x-values that result in even roots of negative numbers.

is defined only for $x \geq 0$. So, its implied domain is the interval $[0, \infty)$. In general, the domain of a function *excludes* values that would cause division by zero *or* result in the even root of a negative number.

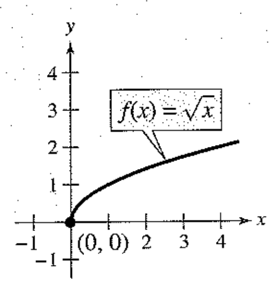
Exploration

Use a graphing utility to graph $y = \sqrt{4 - x^2}$. What is the domain of this function? Then graph $y = \sqrt{x^2 - 4}$. What is the domain of this function? Do the domains of these two functions overlap? If so, for what values?

Library of Functions: Radical Function

Radical functions arise from the use of rational exponents. The most common radical function is the *square root function* given by $f(x) = \sqrt{x}$. The basic characteristics of the square root function are summarized below.

Graph of $f(x) = \sqrt{x}$
Domain: $[0, \infty)$
Range: $[0, \infty)$
Intercept: $(0, 0)$
Increasing on $(0, \infty)$



STUDY TIP

Because the square root function is not defined for $x < 0$, you must be careful when analyzing the domains of complicated functions involving the square root symbol.

Example 5 Finding the Domain of a Function

Find the domain of each function.

a. $f: \{(-3, 0), (-1, 4), (0, 2), (2, 2), (4, -1)\}$

b. $g(x) = -3x^2 + 4x + 5$ **c.** $h(x) = \dfrac{1}{x + 5}$

d. Volume of a sphere: $V = \frac{4}{3}\pi r^3$ **e.** $k(x) = \sqrt{4 - 3x}$

Solution

a. The domain of f consists of all first coordinates in the set of ordered pairs.

Domain $= \{-3, -1, 0, 2, 4\}$

b. The domain of g is the set of all *real* numbers.

c. Excluding x-values that yield zero in the denominator, the domain of h is the set of all real numbers $x \neq -5$.

d. Because this function represents the volume of a sphere, the values of the radius r must be positive. So, the domain is the set of all real numbers r such that $r > 0$.

e. This function is defined only for x-values for which $4 - 3x \geq 0$. By solving this inequality, you will find that the domain of k is all real numbers that are less than or equal to $\frac{4}{3}$.

✓ *Checkpoint* Now try Exercise 51.

STUDY TIP

In Example 5(e), $4 - 3x \geq 0$ is a *linear inequality*. For help with solving linear inequalities, see Appendix E.

In Example 5(d), note that the *domain of a function may be implied by the physical context.* For instance, from the equation $V = \frac{4}{3}\pi r^3$, you would have no reason to restrict r to positive values, but the physical context implies that a sphere cannot have a negative or zero radius.

For some functions, it may be easier to find the domain and range of the function by examining its graph.

Example 6 Finding the Domain and Range of a Function

Use a graphing utility to find the domain and range of the function

$f(x) = \sqrt{9 - x^2}$.

Solution

Graph the function as $y = \sqrt{9 - x^2}$, as shown in Figure 1.16. Using the *trace* feature of a graphing utility, you can determine that the x-values extend from -3 to 3 and the y-values extend from 0 to 3. So, the domain of the function f is all real numbers such that $-3 \leq x \leq 3$ and the range of f is all real numbers such that $0 \leq y \leq 3$.

✓ *Checkpoint* Now try Exercise 61.

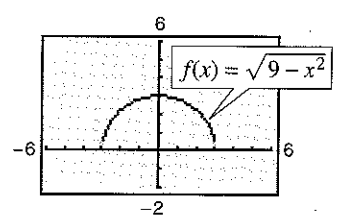


Figure 1.16

Applications

Example 7 Cellular Phone Subscribers

The number N (in millions) of cellular phone subscribers in the United States increased in a linear pattern from 1995 to 1997, as shown in Figure 1.17. Then, in 1998, the number of subscribers took a jump, and until 2001, increased in a *different* linear pattern. These two patterns can be approximated by the function

$$N(t) = \begin{cases} 10.75t - 20.1, & 5 \le t \le 7 \\ 20.11t - 92.8, & 8 \le t \le 11 \end{cases}$$

where t represents the year, with $t = 5$ corresponding to 1995. Use this function to approximate the number of cellular phone subscribers for each year from 1995 to 2001. (Source: Cellular Telecommunications & Internet Association)

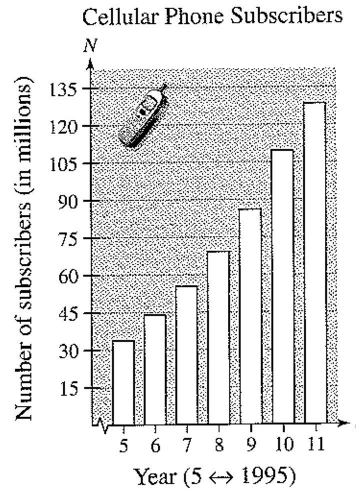


Figure 1.17

Solution

From 1995 to 1997, use $N(t) = 10.75t - 20.1$

$$\underbrace{33.7}_{1995},\ \underbrace{44.4}_{1996},\ \underbrace{55.2}_{1997}$$

From 1998 to 2001, use $N(t) = 20.11t - 92.8$.

$$\underbrace{68.1}_{1998},\ \underbrace{88.2}_{1999},\ \underbrace{108.3}_{2000},\ \underbrace{128.4}_{2001}$$

✓ ***Checkpoint*** Now try Exercise 79.

Example 8 The Path of a Baseball

A baseball is hit at a point 3 feet above the ground at a velocity of 100 feet per second and an angle of 45°. The path of the baseball is given by the function

$$f(x) = -0.0032x^2 + x + 3$$

where y and x are measured in feet. Will the baseball clear a 10-foot fence located 300 feet from home plate?

Algebraic Solution

The height of the baseball is a function of the horizontal distance from home plate. When $x = 300$, you can find the height of the baseball as follows.

$f(x) = -0.0032x^2 + x + 3$ Write original function.

$f(300) = -0.0032(300)^2 + 300 + 3$ Substitute 300 for x.

$= 15$ Simplify.

When $x = 300$, the height of the baseball is 15 feet, so the baseball will clear a 10-foot fence.

✓ ***Checkpoint*** Now try Exercise 81.

Graphical Solution

Use a graphing utility to graph the function $y = -0.0032x^2 + x + 3$. Use the *value* feature or the *zoom* and *trace* features of the graphing utility to estimate that $y = 15$ when $x = 300$, as shown in Figure 1.18. So, the ball will clear a 10-foot fence.

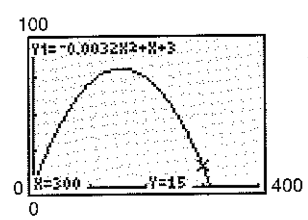


Figure 1.18

Difference Quotients

One of the basic definitions in calculus employs the ratio

$$\frac{f(x+h)-f(x)}{h}, \qquad h \neq 0.$$

This ratio is called a **difference quotient,** as illustrated in Example 9.

Example 9 Evaluating a Difference Quotient

For $f(x) = x^2 - 4x + 7$, find $\dfrac{f(x+h)-f(x)}{h}$.

Solution

$$\frac{f(x+h)-f(x)}{h} = \frac{[(x+h)^2 - 4(x+h) + 7] - (x^2 - 4x + 7)}{h}$$

$$= \frac{x^2 + 2xh + h^2 - 4x - 4h + 7 - x^2 + 4x - 7}{h}$$

$$= \frac{2xh + h^2 - 4h}{h}$$

$$= \frac{h(2x + h - 4)}{h} = 2x + h - 4, \; h \neq 0$$

Checkpoint Now try Exercise 85.

STUDY TIP

Notice in Example 9 that h cannot be zero in the original expression. Therefore, you must restrict the domain of the simplified expression by adding $h \neq 0$ so that the simplified expression is equivalent to the original expression.

Summary of Function Terminology

Function: A **function** is a relationship between two variables such that to each value of the independent variable there corresponds exactly one value of the dependent variable.

Function Notation: $y = f(x)$

- f is the *name* of the function.
- y is the **dependent variable,** or output value.
- x is the **independent variable,** or input value.
- $f(x)$ is the *value of the function at* x.

Domain: The **domain** of a function is the set of all values (inputs) of the independent variable for which the function is defined. If x is in the domain of f, f is said to be *defined* at x. If x is not in the domain of f, f is said to be *undefined* at x.

Range: The **range** of a function is the set of all values (outputs) assumed by the dependent variable (that is, the set of all function values).

Implied Domain: If f is defined by an algebraic expression and the domain is not specified, the **implied domain** consists of all real numbers for which the expression is defined.

The symbol ∫ indicates an example or exercise that highlights algebraic techniques specifically used in calculus.

1.2 Exercises

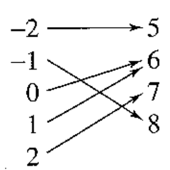
Vocabulary Check

Fill in the blanks.

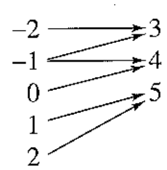
1. A relation that assigns to each element x from a set of inputs, or ________ , exactly one element y in a set of outputs, or ________ , is called a ________ .
2. For an equation that represents y as a function of x, the ________ variable is the set of all x in the domain, and the ________ variable is the set of all y in the range.
3. The function $f(x) = \begin{cases} x^2 - 4, x \le 0 \\ 2x + 1, x > 0 \end{cases}$ is an example of a ________ function.
4. If the domain of the function f is not given, then the set of values of the independent variable for which the expression is defined is called the ________ .
5. In calculus, one of the basic definitions is that of a ________ , given by $\dfrac{f(x+h) - f(x)}{h}$, $h \neq 0$.

In Exercises 1–4, does the relationship describe a function? Explain your reasoning.

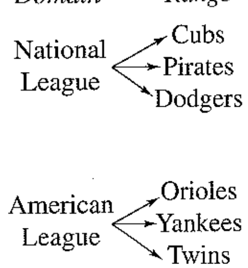
1. *Domain* *Range*



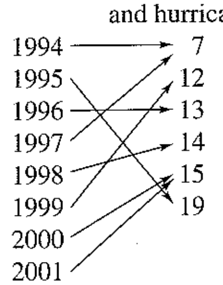
2. *Domain* *Range*



3. *Domain* *Range*



4. *Domain* (Year) *Range* (Number of North Atlantic tropical storms and hurricanes)



In Exercises 5–8, does the table describe a function? Explain your reasoning.

5.

Input Value	-2	-1	0	1	2
Output Value	-8	-1	0	1	8

6.

Input Value	0	1	2	1	0
Output Value	-4	-2	0	2	4

7.

Input Value	10	7	4	7	10
Output Value	3	6	9	12	15

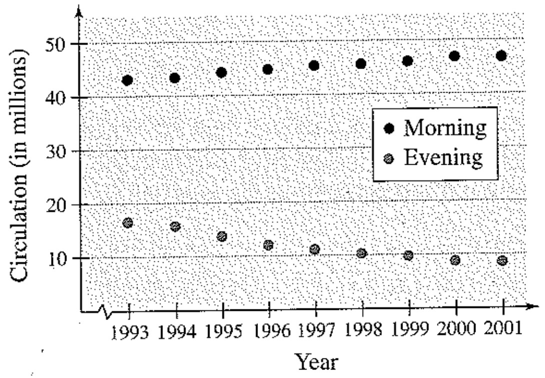
8.

Input Value	0	3	9	12	15
Output Value	3	3	3	3	3

In Exercises 9 and 10, which sets of ordered pairs represent functions from A to B? Explain.

9. $A = \{0, 1, 2, 3\}$ and $B = \{-2, -1, 0, 1, 2\}$
   (a) $\{(0, 1), (1, -2), (2, 0), (3, 2)\}$
   (b) $\{(0, -1), (2, 2), (1, -2), (3, 0), (1, 1)\}$
   (c) $\{(0, 0), (1, 0), (2, 0), (3, 0)\}$
   (d) $\{(0, 2), (3, 0), (1, 1)\}$
10. $A = \{a, b, c\}$ and $B = \{0, 1, 2, 3\}$
   (a) $\{(a, 1), (c, 2), (c, 3), (b, 3)\}$
   (b) $\{(a, 1), (b, 2), (c, 3)\}$
   (c) $\{(1, a), (0, a), (2, c), (3, b)\}$
   (d) $\{(c, 0), (b, 0), (a, 3)\}$

Circulation of Newspapers **In Exercises 11 and 12, use the graph, which shows the circulation (in millions) of daily newspapers in the United States.** (Source: Editor & Publisher Company)



11. Is the circulation of morning newspapers a function of the year? Is the circulation of evening newspapers a function of the year? Explain.

12. Let $f(x)$ represent the circulation of evening newspapers in year x. Find $f(2000)$.

In Exercises 13–24, determine whether the equation represents *y* as a function of *x*.

13. $x^2 + y^2 = 4$ **14.** $x = y^2$

15. $x^2 + y = -1$ **16.** $y = \sqrt{x + 5}$

17. $2x + 3y = 4$ **18.** $x = -y + 5$

19. $y^2 = x^2 - 1$ **20.** $x + y^2 = 3$

21. $y = |4 - x|$ **22.** $|y| = 4 - x$

23. $x = -7$ **24.** $y = 8$

In Exercises 25 and 26, fill in the blanks using the specified function and the given values of the independent variable. Simplify the result.

25. $f(x) = \dfrac{1}{x + 1}$

(a) $f(4) = \dfrac{1}{(\quad) + 1}$

(b) $f(0) = \dfrac{1}{(\quad) + 1}$

(c) $f(4t) = \dfrac{1}{(\quad) + 1}$

(d) $f(x + c) = \dfrac{1}{(\quad) + 1}$

26. $g(x) = x^2 - 2x$

(a) $g(2) = (\quad)^2 - 2(\quad)$

(b) $g(-3) = (\quad)^2 - 2(\quad)$

(c) $g(t + 1) = (\quad)^2 - 2(\quad)$

(d) $g(x + c) = (\quad)^2 - 2(\quad)$

In Exercises 27–38, evaluate the function at each specified value of the independent variable and simplify.

27. $f(x) = 2x - 3$

(a) $f(1)$ (b) $f(-3)$ (c) $f(x - 1)$

28. $g(y) = 7 - 3y$

(a) $g(0)$ (b) $g\left(\frac{7}{3}\right)$ (c) $g(s + 2)$

29. $h(t) = t^2 - 2t$

(a) $h(2)$ (b) $h(1.5)$ (c) $h(x + 2)$

30. $V(r) = \frac{4}{3}\pi r^3$

(a) $V(3)$ (b) $V\left(\frac{3}{2}\right)$ (c) $V(2r)$

31. $f(y) = 3 - \sqrt{y}$

(a) $f(4)$ (b) $f(0.25)$ (c) $f(4x^2)$

32. $f(x) = \sqrt{x + 8} + 2$

(a) $f(-8)$ (b) $f(1)$ (c) $f(x - 8)$

33. $q(x) = \dfrac{1}{x^2 - 9}$

(a) $q(0)$ (b) $q(3)$ (c) $q(y + 3)$

34. $q(t) = \dfrac{2t^2 + 3}{t^2}$

(a) $q(2)$ (b) $q(0)$ (c) $q(-x)$

35. $f(x) = \dfrac{|x|}{x}$

(a) $f(2)$ (b) $f(-2)$ (c) $f(x^2)$

36. $f(x) = |x| + 4$

(a) $f(2)$ (b) $f(-2)$ (c) $f(x^2)$

37. $f(x) = \begin{cases} 2x + 1, & x < 0 \\ 2x + 2, & x \geq 0 \end{cases}$

(a) $f(-1)$ (b) $f(0)$ (c) $f(2)$

38. $f(x) = \begin{cases} x^2 + 2, & x \leq 1 \\ 2x^2 + 2, & x > 1 \end{cases}$

(a) $f(-2)$ (b) $f(1)$ (c) $f(2)$

In Exercises 39–42, complete the table.

39. $h(t) = \frac{1}{2}|t + 3|$

t	-5	-4	-3	-2	-1
$h(t)$					

40. $f(s) = \dfrac{|s - 2|}{s - 2}$

s	0	1	$\frac{3}{2}$	$\frac{5}{2}$	4
$f(s)$					

41. $f(x) = \begin{cases} -\frac{1}{2}x + 4, & x \le 0 \\ (x - 2)^2, & x > 0 \end{cases}$

x	-2	-1	0	1	2
$f(x)$					

42. $h(x) = \begin{cases} 9 - x^2, & x < 3 \\ x - 3, & x \ge 3 \end{cases}$

x	1	2	3	4	5
$h(x)$					

In Exercises 43–46, find all real values of x such that $f(x) = 0$.

43. $f(x) = 15 - 3x$

44. $f(x) = 5x + 1$

45. $f(x) = \dfrac{3x - 4}{5}$

46. $f(x) = \dfrac{12 - x^2}{5}$

In Exercises 47 and 48, find the value(s) of x for which $f(x) = g(x)$.

47. $f(x) = x^2, \quad g(x) = x + 2$

48. $f(x) = x^2 + 2x + 1, \quad g(x) = 3x + 3$

In Exercises 49–58, find the domain of the function.

49. $f(x) = 5x^2 + 2x - 1$

50. $g(x) = 1 - 2x^2$

51. $h(t) = \dfrac{4}{t}$

52. $s(y) = \dfrac{3y}{y + 5}$

53. $f(x) = \sqrt[3]{x - 4}$

54. $f(x) = \sqrt[4]{x^2 + 3x}$

55. $g(x) = \dfrac{1}{x} - \dfrac{3}{x + 2}$

56. $h(x) = \dfrac{10}{x^2 - 2x}$

57. $g(y) = \dfrac{y + 2}{\sqrt{y - 10}}$

58. $f(x) = \dfrac{\sqrt{x + 6}}{6 + x}$

In Exercises 59–62, use a graphing utility to graph the function. Find the domain and range of the function.

59. $f(x) = \sqrt{4 - x^2}$

60. $f(x) = \sqrt{x^2 + 1}$

61. $g(x) = |2x + 3|$

62. $g(x) = |x - 5|$

In Exercises 63–66, assume that the domain of f is the set $A = \{-2, -1, 0, 1, 2\}$. Determine the set of ordered pairs representing the function f.

63. $f(x) = x^2$

64. $f(x) = x^2 - 3$

65. $f(x) = |x| + 2$

66. $f(x) = |x + 1|$

67. ***Geometry*** Write the area A of a circle as a function of its circumference C.

68. ***Geometry*** Write the area A of an equilateral triangle as a function of the length s of its sides.

69. ***Exploration*** The cost per unit to produce a radio model is \$60. The manufacturer charges \$90 per unit for orders of 100 or less. To encourage large orders, the manufacturer reduces the charge by \$0.15 per radio for each unit ordered in excess of 100 (for example, there would be a charge of \$87 per radio for an order size of 120).

(a) The table shows the profit P (in dollars) for various numbers of units ordered, x. Use the table to estimate the maximum profit.

Units, x	Profit, P
110	3135
120	3240
130	3315
140	3360
150	3375
160	3360
170	3315

(b) Plot the points (x, P) from the table in part (a). Does the relation defined by the ordered pairs represent P as a function of x?

(c) If P is a function of x, write the function and determine its domain.

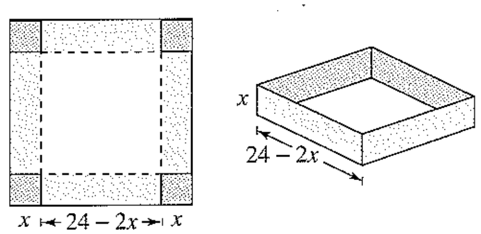
70. ***Exploration*** An open box of maximum volume is to be made from a square piece of material, 24 centimeters on a side, by cutting equal squares from the corners and turning up the sides (see figure).

(a) The table shows the volume V (in cubic centimeters) of the box for various heights x (in centimeters). Use the table to estimate the maximum volume.

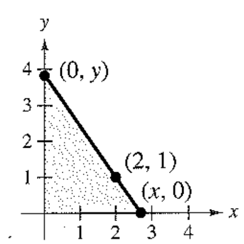
Height, x	Volume, V
1	484
2	800
3	972
4	1024
5	980
6	864

(b) Plot the points (x, V) from the table in part (a). Does the relation defined by the ordered pairs represent V as a function of x?

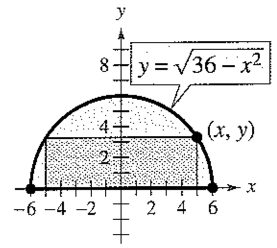
(c) If V is a function of x, write the function and determine its domain.



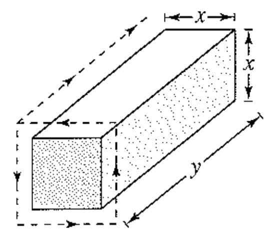
71. ***Geometry*** A right triangle is formed in the first quadrant by the x- and y-axes and a line through the point $(2, 1)$ (see figure). Write the area A of the triangle as a function of x, and determine the domain of the function.



72. ***Geometry*** A rectangle is bounded by the x-axis and the semicircle $y = \sqrt{36 - x^2}$ (see figure). Write the area A of the rectangle as a function of x, and determine the domain of the function.



73. ***Postal Regulations*** A rectangular package to be sent by the U.S. Postal Service can have a maximum combined length and girth (perimeter of a cross section) of 108 inches (see figure).



(a) Write the volume V of the package as a function of x.

(b) What is the domain of the function?

(c) Use a graphing utility to graph the function. Be sure to use the appropriate viewing window.

(d) What dimensions will maximize the volume of the package? Explain.

74. ***Cost, Revenue, and Profit*** A company produces a toy for which the variable cost is \$12.30 per unit and the fixed costs are \$98,000. The toy sells for \$17.98. Let x be the number of units produced and sold.

(a) The total cost for a business is the sum of the variable cost and the fixed costs. Write the total cost C as a function of the number of units produced.

(b) Write the revenue R as a function of the number of units sold.

(c) Write the profit P as a function of the number of units sold. (*Note:* $P = R - C$.)

Revenue **In Exercises 75–78, use the table, which shows the monthly revenue y (in thousands of dollars) of a landscaping business for each month of 2003, with $x = 1$ representing January.**

Month, x	Revenue, y
1	5.2
2	5.6
3	6.6
4	8.3
5	11.5
6	15.8
7	12.8
8	10.1
9	8.6
10	6.9
11	4.5
12	2.7

A mathematical model that represents this data is

$$f(x) = \begin{cases} -1.97x + 26.3 \\ 0.505x^2 - 1.47x + 6.3 \end{cases}$$

75. What is the domain of each part of the piecewise-defined function? Explain your reasoning.

76. Use the mathematical model to find $f(5)$. Interpret your results in the context of the problem.

77. Use the mathematical model to find $f(11)$. Interpret your results in the context of the problem.

78. How do the values obtained from the model in Exercises 76 and 77 compare with the actual data values?

79. ***Motor Vehicles*** The number n (in billions) of miles traveled by vans, pickup trucks, and sport utility vehicles in the United States from 1990 to 2000 can be approximated by the model

$$n(t) = \begin{cases} -9.2t^2 + 84.5t + 575, & 0 \le t \le 4 \\ 26.8t + 657, & 5 \le t \le 10 \end{cases}$$

where t represents the year, with $t = 0$ corresponding to 1990. Use the *table* feature of a graphing utility to approximate the number of miles traveled by vans, pickup trucks, and sport utility vehicles for each year from 1990 to 2000. (Source: U.S. Federal Highway Administration)

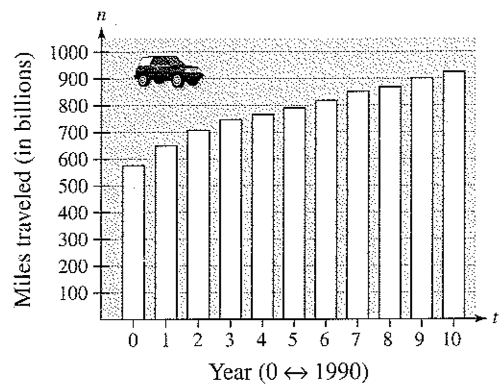


Figure for 79

80. ***Transportation*** For groups of 80 or more people, a charter bus company determines the rate per person according to the formula

Rate $= 8 - 0.05(n - 80), \quad n \ge 80$

where the rate is given in dollars and n is the number of people.

(a) Write the revenue R of the bus company as a function of n.

(b) Use the function from part (a) to complete the table. What can you conclude?

n	90	100	110	120	130	140	150
$R(n)$							

(c) Use a graphing utility to graph R and determine the number of people that will produce a maximum revenue. Compare the result with your conclusion from part (b).

81. ***Physics*** The force F (in tons) of water against the face of a dam is estimated by the function $F(y) = 149.76\sqrt{10}y^{5/2}$, where y is the depth of the water in feet.

(a) Complete the table. What can you conclude from the table?

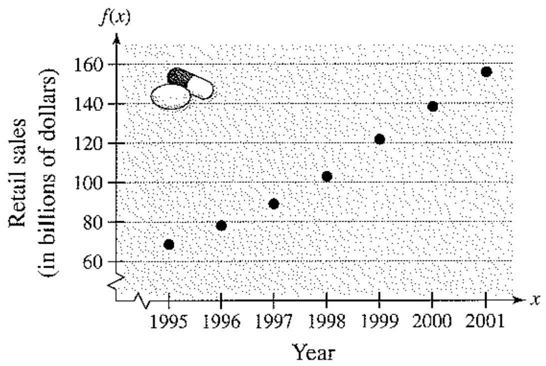
y	5	10	20	30	40
$F(y)$					

(b) Use a graphing utility to graph the function. Describe your viewing window.

(c) Use the table to approximate the depth at which the force against the dam is 1,000,000 tons. How could you find a better estimate?

(d) Verify your answer in part (c) graphically.

82. ***Data Analysis*** The graph shows the retail sales (in billions of dollars) of prescription drugs in the United States from 1995 through 2001. Let $f(x)$ represent the retail sales in year x. (Source: National Association of Chain Drug Stores)



(a) Find $f(1998)$.

(b) Find $\dfrac{f(2001) - f(1995)}{2001 - 1995}$

and interpret the result in the context of the problem.

(c) An approximate model for the function is

$P(t) = -0.1556t^3 + 4.657t^2 - 28.75t + 115.7,$
$5 \le t \le 11$

where P is the retail sales (in billions of dollars) and t represents the year, with $t = 5$ corresponding to 1995. Complete the table and compare the results with the data.

t	5	6	7	8	9	10	11
$P(t)$							

(d) Use a graphing utility to graph the model and data in the same viewing window. Comment on the validity of the model.

In Exercises 83–88, find the difference quotient and simplify your answer.

83. $f(x) = 2x, \quad \dfrac{f(x + c) - f(x)}{c}, \quad c \ne 0$

84. $g(x) = 3x - 1, \quad \dfrac{g(x + h) - g(x)}{h}, \quad h \ne 0$

85. $f(x) = x^2 - x + 1, \quad \dfrac{f(2 + h) - f(2)}{h}, \quad h \ne 0$

86. $f(x) = x^3 + x, \quad \dfrac{f(x + h) - f(x)}{h}, \quad h \ne 0$

87. $f(t) = \dfrac{1}{t}, \quad \dfrac{f(t) - f(1)}{t - 1}, \quad t \ne 1$

88. $f(x) = \dfrac{4}{x + 1}, \quad \dfrac{f(x) - f(7)}{x - 7}, \quad x \ne 7$

Synthesis

True or False? **In Exercises 89 and 90, determine whether the statement is true or false. Justify your answer.**

89. The domain of the function $f(x) = x^4 - 1$ is $(-\infty, \infty)$, and the range of $f(x)$ is $(0, \infty)$.

90. The set of ordered pairs $\{(-8, -2), (-6, 0), (-4, 0), (-2, 2), (0, 4), (2, -2)\}$ represents a function.

Exploration **In Exercises 91 and 92, match the data with one of the functions $g(x) = cx^2$ or $r(x) = c/x$ and determine the value of the constant c such that the function fits the data given in the table.**

91.

x	-4	-1	0	1	4
y	-8	-32	Undef.	32	8

92.

x	-4	-1	0	1	4
y	-32	-2	0	-2	-32

93. ***Writing*** In your own words, explain the meanings of *domain* and *range*.

94. ***Think About It*** Describe an advantage of function notation.

Review

In Exercises 95–98, perform the operations and simplify.

95. $12 - \dfrac{4}{x + 2}$

96. $\dfrac{3}{x^2 + x - 20} + \dfrac{x}{x^2 + 4x - 5}$

97. $\dfrac{2x^3 + 11x^2 - 6x}{5x} \cdot \dfrac{x + 10}{2x^2 + 5x - 3}$

98. $\dfrac{x + 7}{2(x - 9)} \div \dfrac{x - 7}{2(x - 9)}$

The symbol $\int$ indicates an example or exercise that highlights algebraic techniques specifically used in calculus.

1.3 Graphs of Functions

The Graph of a Function

In Section 1.2, functions were represented graphically by points on a graph in a coordinate plane in which the input values are represented by the horizontal axis and the output values are represented by the vertical axis. The **graph of a function** f is the collection of ordered pairs $(x, f(x))$ such that x is in the domain of f. As you study this section, remember the geometric interpretations of x and $f(x)$.

$x =$ the directed distance from the y-axis

$f(x) =$ the directed distance from the x-axis

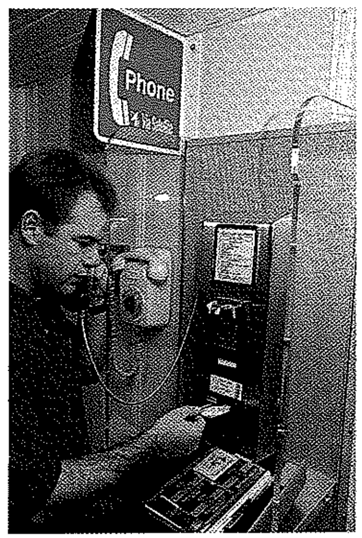
Example 1 shows how to use the graph of a function to find the domain and range of the function.

What you should learn

- Find the domains and ranges of functions and use the Vertical Line Test for functions.
- Determine intervals on which functions are increasing, decreasing, or constant.
- Determine relative maximum and relative minimum values of functions.
- Identify and graph step functions and other piecewise-defined functions.
- Identify even and odd functions.

Why you should learn it

Graphs of functions provide a visual relationship between two variables. Exercise 81 on page 40 shows how the graph of a step function can represent the cost of a telephone call.



Jeff Greenberg/Peter Arnold, Inc.

Example 1 Finding the Domain and Range of a Function

Use the graph of the function f shown in Figure 1.19 to find (a) the domain of f, (b) the function values $f(-1)$ and $f(2)$, and (c) the range of f.

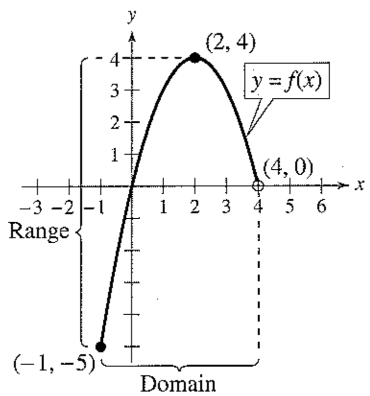


Figure 1.19

Solution

a. The closed dot at $(-1, -5)$ indicates that $x = -1$ is in the domain of f, whereas the open dot at $(4, 0)$ indicates that $x = 4$ is not in the domain. So, the domain of f is all x in the interval $[-1, 4)$.

b. Because $(-1, -5)$ is a point on the graph of f, it follows that

$$f(-1) = -5.$$

Similarly, because $(2, 4)$ is a point on the graph of f, it follows that

$$f(2) = 4.$$

c. Because the graph does not extend below $f(-1) = -5$ or above $f(2) = 4$, the range of f is the interval $[-5, 4]$.

✓ *Checkpoint* Now try Exercise 3.

STUDY TIP

The use of dots (open or closed) at the extreme left and right points of a graph indicates that the graph does not extend beyond these points. If no such dots are shown, assume that the graph extends beyond these points.

Example 2 Finding the Domain and Range of a Function

Find the domain and range of

$f(x) = \sqrt{x - 4}$.

Algebraic Solution

Because the expression under a radical cannot be negative, the domain of $f(x) = \sqrt{x - 4}$ is the set of all real numbers such that $x - 4 \geq 0$. Solve this linear inequality for x as follows. (For help with solving linear inequalities, see Appendix E.)

$x - 4 \geq 0$ Write original inequality.

$x \geq 4$ Add 4 to each side.

So, the domain is the set of all real numbers greater than or equal to 4. Because the value of a radical expression is never negative, the range of $f(x) = \sqrt{x - 4}$ is the set of all nonnegative real numbers.

✓ *Checkpoint* Now try Exercise 7.

Graphical Solution

Use a graphing utility to graph the equation $y = \sqrt{x - 4}$, as shown in Figure 1.20. Use the *trace* feature to determine that the x-coordinates of points on the graph extend from 4 to the right. When x is greater than or equal to 4, the expression under the radical is nonnegative. So, you can conclude that the domain is the set of all real numbers greater than or equal to 4. From the graph, you can see that the y-coordinates of points on the graph extend from 0 upwards. So you can estimate the range to be the set of all nonnegative real numbers.

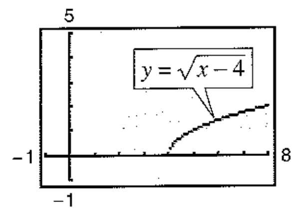


Figure 1.20

By the definition of a function, at most one y-value corresponds to a given x-value. It follows, then, that a vertical line can intersect the graph of a function at most once. This leads to the **Vertical Line Test** for functions.

> **Vertical Line Test for Functions**
>
> A set of points in a coordinate plane is the graph of y as a function of x if and only if no vertical line intersects the graph at more than one point.

Example 3 Vertical Line Test for Functions

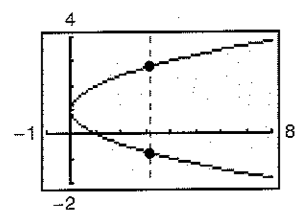
Use the Vertical Line Test to decide whether the graphs in Figure 1.21 represent y as a function of x.

Solution

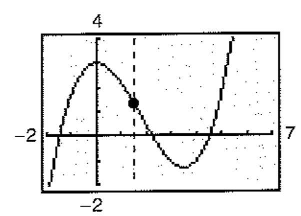
a. This is *not* a graph of y as a function of x because you can find a vertical line that intersects the graph twice.

b. This *is* a graph of y as a function of x because every vertical line intersects the graph at most once.

✓ *Checkpoint* Now try Exercise 13.



(a)



(b)

Figure 1.21

Increasing and Decreasing Functions

The more you know about the graph of a function, the more you know about the function itself. Consider the graph shown in Figure 1.22. Moving from *left to right*, this graph falls from $x = -2$ to $x = 0$, is constant from $x = 0$ to $x = 2$, and rises from $x = 2$ to $x = 4$.

> **Increasing, Decreasing, and Constant Functions**
>
> A function f is **increasing** on an interval if, for any x_1 and x_2 in the interval,
>
> $x_1 < x_2$ implies $f(x_1) < f(x_2)$.
>
> A function f is **decreasing** on an interval if, for any x_1 and x_2 in the interval,
>
> $x_1 < x_2$ implies $f(x_1) > f(x_2)$.
>
> A function f is **constant** on an interval if, for any x_1 and x_2 in the interval,
>
> $f(x_1) = f(x_2)$.

TECHNOLOGY TIP

Most graphing utilities are designed to graph functions of x more easily than other types of equations. For instance, the graph shown in Figure 1.21(a) represents the equation $x - (y - 1)^2 = 0$. To use a graphing utility to duplicate this graph you must first solve the equation for y to obtain $y = 1 \pm \sqrt{x}$, and then graph the two equations $y_1 = 1 + \sqrt{x}$ and $y_2 = 1 - \sqrt{x}$ in the same viewing window.

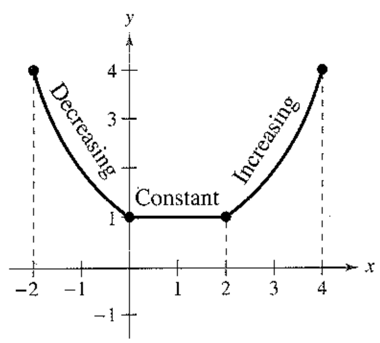


Figure 1.22

Example 4 Increasing and Decreasing Functions

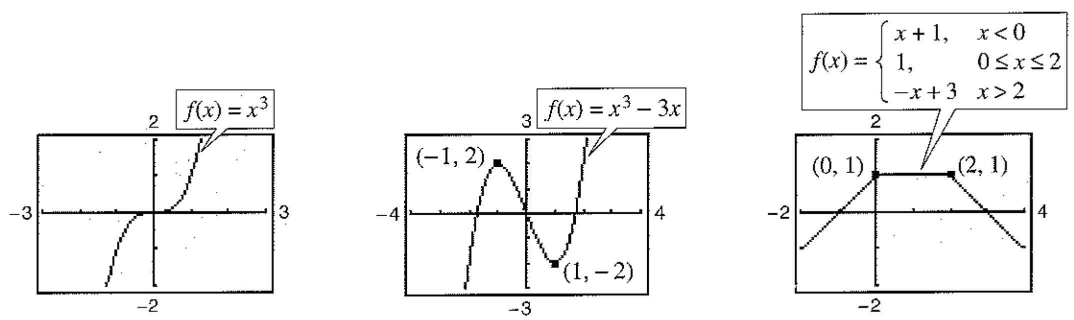
In Figure 1.23, determine the open intervals on which each function is increasing, decreasing, or constant.

Solution

a. Although it might appear that there is an interval in which this function is constant, you can see that if $x_1 < x_2$, then $(x_1)^3 < (x_2)^3$, which implies that $f(x_1) < f(x_2)$. So, the function is increasing over the entire real line.

b. This function is increasing on the interval $(-\infty, -1)$, decreasing on the interval $(-1, 1)$, and increasing on the interval $(1, \infty)$.

c. This function is increasing on the interval $(-\infty, 0)$, constant on the interval $[0, 2]$, and decreasing on the interval $(2, \infty)$.



(a) (b) (c)

Figure 1.23

✓ *Checkpoint* Now try Exercise 19.

Relative Minimum and Maximum Values

The points at which a function changes its increasing, decreasing, or constant behavior are helpful in determining the relative maximum or relative minimum values of the function.

Definition of Relative Minimum and Relative Maximum

A function value $f(a)$ is called a **relative minimum** of f if there exists an interval (x_1, x_2) that contains a such that

$$x_1 < x < x_2 \quad \text{implies} \quad f(a) \le f(x).$$

A function value $f(a)$ is called a **relative maximum** of f if there exists an interval (x_1, x_2) that contains a such that

$$x_1 < x < x_2 \quad \text{implies} \quad f(a) \ge f(x).$$

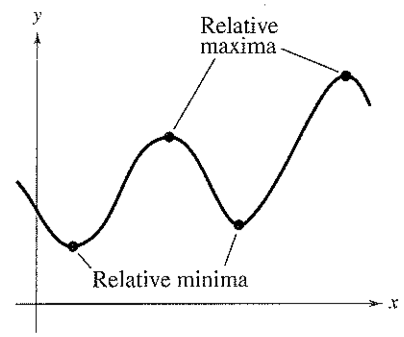


Figure 1.24

Figure 1.24 shows several different examples of relative minima and relative maxima. In Section 2.1, you will study a technique for finding the *exact points* at which a second-degree polynomial function has a relative minimum or relative maximum. For the time being, however, you can use a graphing utility to find reasonable approximations of these points.

Example 5 Approximating a Relative Minimum

Use a graphing utility to approximate the relative minimum of the function given by $f(x) = 3x^2 - 4x - 2$.

Solution

The graph of f is shown in Figure 1.25. By using the *zoom* and *trace* features of a graphing utility, you can estimate that the function has a relative minimum at the point

$(0.67, -3.33)$. See Figure 1.26.

Later, in Section 2.1, you will be able to determine that the exact point at which the relative minimum occurs is $\left(\frac{2}{3}, -\frac{10}{3}\right)$.

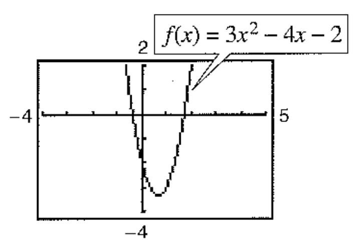


Figure 1.25

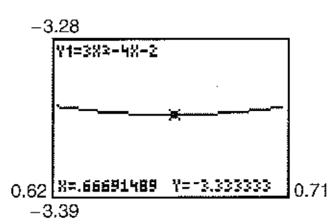


Figure 1.26

✓ ***Checkpoint*** Now try Exercise 29.

TECHNOLOGY TIP Some graphing utilities have built-in programs that will find minimum or maximum values. These features are demonstrated in Example 6.

TECHNOLOGY TIP

When you use a graphing utility to estimate the x- and y-values of a relative minimum or relative maximum, the *zoom* feature will often produce graphs that are nearly flat, as shown in Figure 1.26. To overcome this problem, you can manually change the vertical setting of the viewing window. The graph will vertically stretch if the values of Ymin and Ymax are closer together.

Example 6 Approximating Relative Minima and Maxima

Use a graphing utility to approximate the relative minimum and relative maximum of the function given by $f(x) = -x^3 + x$.

Solution

The graph of f is shown in Figure 1.27. By using the *zoom* and *trace* features or the *minimum* and *maximum* features of the graphing utility, you can estimate that the function has a relative minimum at the point

$(-0.58, -0.38)$ See Figure 1.28.

and a relative maximum at the point

$(0.58, 0.38)$. See Figure 1.29.

If you take a course in calculus, you will learn a technique for finding the exact points at which this function has a relative minimum and a relative maximum.

Checkpoint Now try Exercise 31.

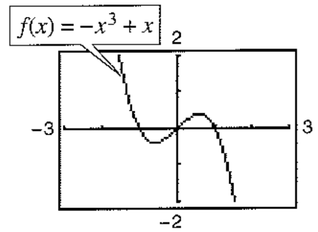


Figure 1.27

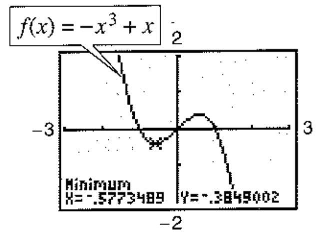


Figure 1.28

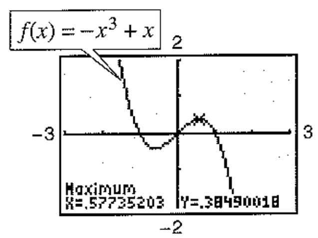


Figure 1.29

Example 7 Temperature

During a 24-hour period, the temperature y (in degrees Fahrenheit) of a certain city can be approximated by the model

$$y = 0.026x^3 - 1.03x^2 + 10.2x + 34, \qquad 0 \le x \le 24$$

where x represents the time of day, with $x = 0$ corresponding to 6 A.M. Approximate the maximum and minimum temperatures during this 24-hour period.

Solution

To solve this problem, graph the function as shown in Figure 1.30. Using the *zoom* and *trace* features or the *maximum* feature of a graphing utility, you can determine that the maximum temperature during the 24-hour period was approximately 64°F. This temperature occurred at about 12:36 P.M. ($x \approx 6.6$), as shown in Figure 1.31. Using the *zoom* and *trace* features or the *minimum* feature, you can determine that the minimum temperature during the 24-hour period was approximately 34°F, which occurred at about 1:48 A.M. ($x \approx 19.8$), as shown in Figure 1.32.

TECHNOLOGY SUPPORT

For instructions on how to use the *minimum* and *maximum* features, see Appendix A; for specific keystrokes, go to the text website at *college.hmco.com*.

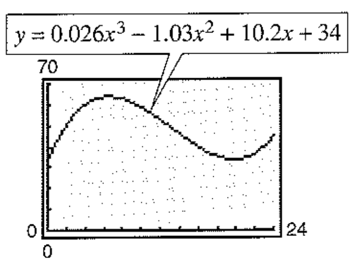


Figure 1.30

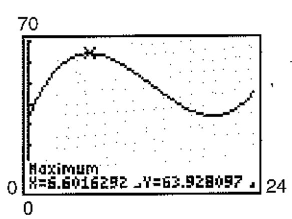


Figure 1.31

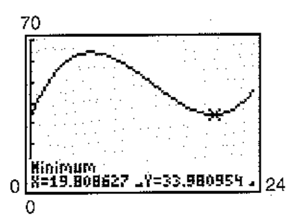


Figure 1.32

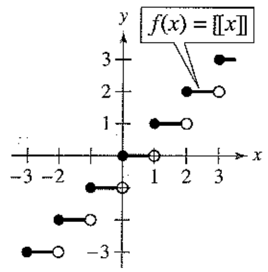
Checkpoint Now try Exercise 87.

Graphing Step Functions and Piecewise-Defined Functions

Library of Functions: Greatest Integer Function

The *greatest integer function*, denoted by $[\![x]\!]$ and defined as the greatest integer less than or equal to x, has an infinite number of breaks or steps—one at each integer value in its domain. The basic characteristics of the greatest integer function are summarized below.

Graph of $f(x) = [\![x]\!]$
Domain: $(-\infty, \infty)$
Range: the set of integers
x-intercepts: in the interval $[0, 1)$
y-intercept: $(0, 0)$
Constant between each pair of consecutive integers
Jumps vertically one unit at each integer value



Could you describe the greatest integer function using a piecewise-defined function? How does the graph of the greatest integer function differ from the graph of a line with a slope of zero?

TECHNOLOGY TIP

Most graphing utilities display graphs in *connected mode*, which means that the graph has no breaks. When you are sketching graphs that do have breaks, it is better to use *dot mode*. Graph the greatest integer function [often called Int (x)] in connected and dot modes, and compare the two results.

Because of the vertical jumps described above, the greatest integer function is an example of a **step function** whose graph resembles a set of stairsteps. Some values of the greatest integer function are as follows.

$$[\![-1]\!] = (\text{greatest integer} \leq -1) = -1$$

$$[\![\tfrac{1}{10}]\!] = (\text{greatest integer} \leq \tfrac{1}{10}) = 0$$

$$[\![1.5]\!] = (\text{greatest integer} \leq 1.5) = 1$$

In Section 1.2, you learned that a piecewise-defined function is a function that is defined by two or more equations over a specified domain. To sketch the graph of a piecewise-defined function, you need to sketch the graph of each equation on the appropriate portion of the domain.

Example 8 Graphing a Piecewise-Defined Function

Sketch the graph of $f(x) = \begin{cases} 2x + 3, & x \leq 1 \\ -x + 4, & x > 1 \end{cases}$ by hand.

Solution

This piecewise-defined function is composed of two linear functions. At and to the left of $x = 1$, the graph is the line given by $y = 2x + 3$. To the right of $x = 1$, the graph is the line given by $y = -x + 4$ (see Figure 1.33). Notice that the point $(1, 5)$ is a solid dot and the point $(1, 3)$ is an open dot. This is because $f(1) = 5$.

Checkpoint Now try Exercise 41.

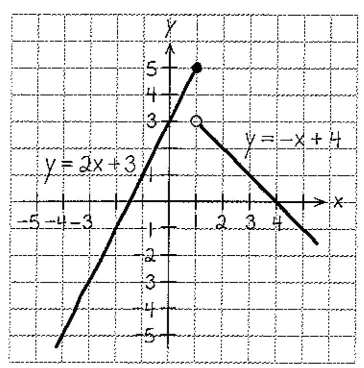
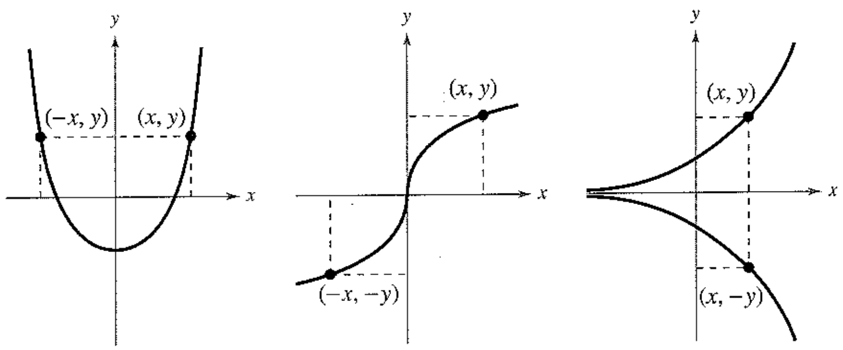


Figure 1.33

Even and Odd Functions

A graph has *symmetry with respect to the y-axis* if whenever (x, y) is on the graph, so is the point $(-x, y)$. A graph has *symmetry with respect to the origin* if whenever (x, y) is on the graph, so is the point $(-x, -y)$. A graph has *symmetry with respect to the x-axis* if whenever (x, y) is on the graph, so is the point $(x, -y)$. A function whose graph is symmetric with respect to the y-axis is an **even** function. A function whose graph is symmetric with respect to the origin is an **odd** function. A graph that is symmetric with respect to the x-axis is not the graph of a function (except for the graph of $y = 0$). These three types of symmetry are illustrated in Figure 1.34.



Symmetric to y-axis
Even function

Symmetric to origin
Odd function

Symmetric to x-axis
Not a function

Figure 1.34

> **Test for Even and Odd Functions**
> A function f is **even** if, for each x in the domain of f, $f(-x) = f(x)$.
> A function f is **odd** if, for each x in the domain of f, $f(-x) = -f(x)$.

Example 9 Testing for Evenness and Oddness

Is the function given by $f(x) = |x|$ even, odd, or neither?

Algebraic Solution

This function is even because

$$\begin{aligned} f(-x) &= |-x| \\ &= |x| \\ &= f(x). \end{aligned}$$

✓ ***Checkpoint*** Now try Exercise 49.

Graphical Solution

Use a graphing utility to enter $y = |x|$ in the equation editor, as shown in Figure 1.35. Then graph the function using a standard viewing window, as shown in Figure 1.36. You can see that the graph appears to be symmetric about the y-axis. So, the function is even.

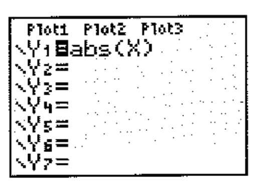


Figure 1.35

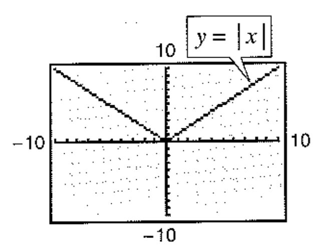


Figure 1.36

Example 10 Even and Odd Functions

Determine whether each function is even, odd, or neither.

a. $g(x) = x^3 - x$

b. $h(x) = x^2 + 1$

c. $f(x) = x^3 - 1$

Algebraic Solution

a. This function is odd because

$$\begin{aligned} g(-x) &= (-x)^3 - (-x) \\ &= -x^3 + x \\ &= -(x^3 - x) \\ &= -g(x). \end{aligned}$$

b. This function is even because

$$\begin{aligned} h(-x) &= (-x)^2 + 1 \\ &= x^2 + 1 \\ &= h(x). \end{aligned}$$

c. Substituting $-x$ for x produces

$$\begin{aligned} f(-x) &= (-x)^3 - 1 \\ &= -x^3 - 1. \end{aligned}$$

Because $f(x) = x^3 - 1$ and $-f(x) = -x^3 + 1$, you can conclude that $f(-x) \neq f(x)$ and $f(-x) \neq -f(x)$. So, the function is neither even nor odd.

✓ ***Checkpoint*** Now try Exercise 51.

Graphical Solution

a. In Figure 1.37, the graph is symmetric with respect to the origin. So, this function is odd.

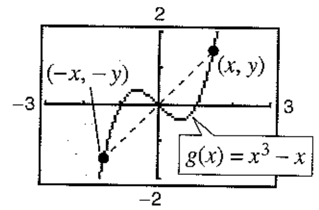


Figure 1.37

b. In Figure 1.38, the graph is symmetric with respect to the y-axis. So, this function is even.

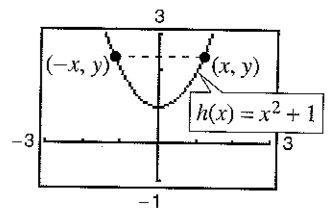


Figure 1.38

c. In Figure 1.39, the graph is neither symmetric with respect to the origin nor with respect to the y-axis. So, this function is neither even nor odd.

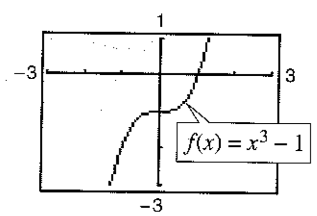


Figure 1.39

To help visualize symmetry with respect to the origin, place a pin at the origin of a graph and rotate the graph 180°. If the result after rotation coincides with the original graph, the graph is symmetric with respect to the origin.

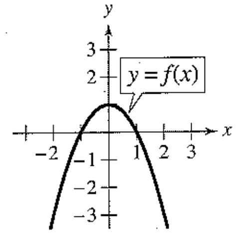
1.3 Exercises

Vocabulary Check

Fill in the blanks.

1. The graph of a function f is a collection of ________ (x, y) such that x is in the domain of f.
2. The ________ is used to determine whether the graph of an equation is a function of y in terms of x.
3. A function f is ________ on an interval if, for any x_1 and x_2 in the interval, $x_1 < x_2$ implies $f(x_1) > f(x_2)$.
4. A function value $f(a)$ is a relative ________ of f if there exists an interval (x_1, x_2) containing a such that $x_1 < x < x_2$ implies $f(a) \leq f(x)$.
5. The function $f(x) = [\![x]\!]$ is called the ________ function, and is an example of a step function.
6. A function f is ________ if, for each x in the domain of f, $f(-x) = f(x)$.

In Exercises 1–4, use the graph of the function to find the domain and range of f. Then find $f(0)$.

1.


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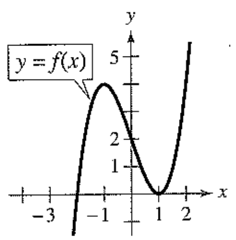

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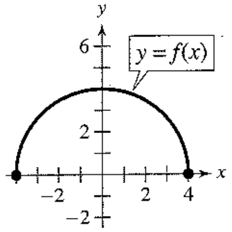

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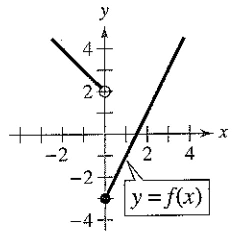
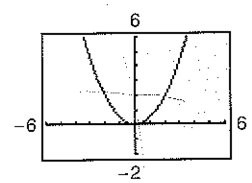
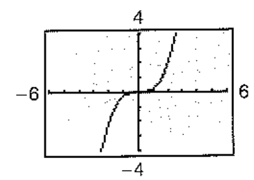


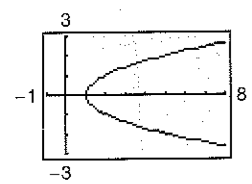
In Exercises 5–10, use a graphing utility to graph the function and estimate its domain and range. Then find the domain and range algebraically.

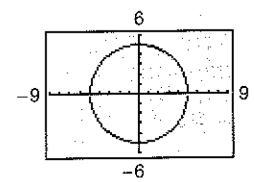
5. $f(x) = 2x^2 + 3$
6. $f(x) = -x^2 - 1$
7. $f(x) = \sqrt{x - 1}$
8. $h(t) = \sqrt{4 - t^2}$
9. $f(x) = |x + 3|$
10. $f(x) = -\frac{1}{4}|x - 5|$

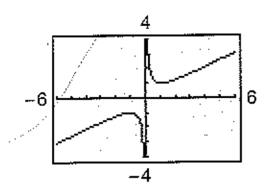
In Exercises 11–16, use the Vertical Line Test to determine whether y is a function of x. Describe how you can use a graphing utility to produce the given graph.

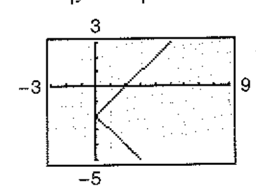
11. $y = \frac{1}{2}x^2$


12. $y = \frac{1}{4}x^3$


13. $x - y^2 = 1$


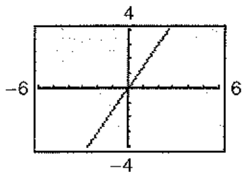
14. $x^2 + y^2 = 25$


15. $x^2 = 2xy - 1$


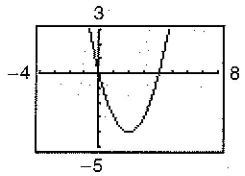
16. $x = |y + 2|$


In Exercises 17–20, determine the intervals over which the function is increasing, decreasing, or constant.

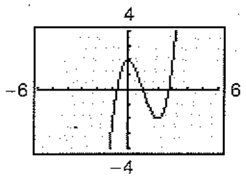
17. $f(x) = \frac{3}{2}x$



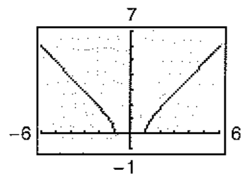
18. $f(x) = x^2 - 4x$



19. $f(x) = x^3 - 3x^2 + 2$



20. $f(x) = \sqrt{x^2 - 1}$



In Exercises 21–28, (a) use a graphing utility to graph the function and (b) determine the open intervals on which the function is increasing, decreasing, or constant.

21. $f(x) = 3$ **22.** $f(x) = x$

23. $f(x) = x^{2/3}$ **24.** $f(x) = -x^{3/4}$

25. $f(x) = x\sqrt{x + 3}$ **26.** $f(x) = \sqrt{1 - x}$

27. $f(x) = |x + 1| + |x - 1|$

28. $f(x) = -|x + 4| - |x + 1|$

In Exercises 29–34, use a graphing utility to approximate (to two decimal places) any relative minimum or maximum values of the function.

29. $f(x) = x^2 - 6x$ **30.** $f(x) = 3x^2 - 2x - 5$

31. $y = 2x^3 + 3x^2 - 12x$ **32.** $y = x^3 - 6x^2 + 15$

33. $h(x) = (x - 1)\sqrt{x}$ **34.** $g(x) = x\sqrt{4 - x}$

In Exercises 35–40, (a) approximate the relative minimum or maximum values of the function by sketching its graph using the point-plotting method, (b) use a graphing utility to approximate (to two decimal places) any relative minimum or maximum values, and (c) compare your answers from parts (a) and (b).

35. $f(x) = x^2 - 4x - 5$ **36.** $f(x) = 3x^2 - 12$

37. $f(x) = x^3 - 8x$ **38.** $f(x) = -x^3 + 7x$

39. $f(x) = (x - 4)^{2/3}$ **40.** $f(x) = \sqrt{4x^2 + 1}$

In Exercises 41–48, sketch the graph of the piecewise-defined function by hand.

41. $f(x) = \begin{cases} 2x + 3, & x < 0 \\ 3 - x, & x \geq 0 \end{cases}$

42. $f(x) = \begin{cases} x + 6, & x \leq -4 \\ 2x - 4, & x > -4 \end{cases}$

43. $f(x) = \begin{cases} \sqrt{4 + x}, & x < 0 \\ \sqrt{4 - x}, & x \geq 0 \end{cases}$

44. $f(x) = \begin{cases} 1 - (x - 1)^2, & x \leq 2 \\ \sqrt{x - 2}, & x > 2 \end{cases}$

45. $f(x) = \begin{cases} x + 3, & x \leq 0 \\ 3, & 0 < x \leq 2 \\ 2x - 1, & x > 2 \end{cases}$

46. $g(x) = \begin{cases} x + 5, & x \leq -3 \\ -2, & -3 < x < 1 \\ 5x - 4, & x \geq 1 \end{cases}$

47. $f(x) = \begin{cases} 2x + 1, & x \leq -1 \\ x^2 - 2, & x > -1 \end{cases}$

48. $h(x) = \begin{cases} 3 + x, & x < 0 \\ x^2 + 1, & x \geq 0 \end{cases}$

In Exercises 49–56, algebraically determine whether the function is even, odd, or neither. Verify your answer using a graphing utility.

49. $f(t) = t^2 + 2t - 3$ **50.** $f(x) = x^6 - 2x^2 + 3$

51. $g(x) = x^3 - 5x$ **52.** $h(x) = x^3 - 5$

53. $f(x) = x\sqrt{1 - x^2}$ **54.** $f(x) = x\sqrt{x + 5}$

55. $g(s) = 4s^{2/3}$ **56.** $f(s) = 4s^{3/2}$

Think About It **In Exercises 57–62, find the coordinates of a second point on the graph of a function *f* if the given point is on the graph and the function is (a) even and (b) odd.**

57. $\left(-\frac{3}{2}, 4\right)$ **58.** $\left(-\frac{5}{3}, -7\right)$

59. $(4, 9)$ **60.** $(5, -1)$

61. $(x, -y)$ **62.** $(2a, 2c)$

In Exercises 63–72, use a graphing utility to graph the function and determine whether it is even, odd, or neither. Verify your answer algebraically.

63. $f(x) = 5$ **64.** $f(x) = -9$

65. $f(x) = 3x - 2$ **66.** $f(x) = 5 - 3x$

67. $h(x) = x^2 - 4$ **68.** $f(x) = -x^2 - 8$

69. $f(x) = \sqrt{1 - x}$

70. $g(t) = \sqrt[3]{t - 1}$

71. $f(x) = |x + 2|$

72. $f(x) = -|x - 5|$

In Exercises 73–76, graph the function and determine the interval(s) (if any) on the real axis for which $f(x) \geq 0$. Use a graphing utility to verify your results.

73. $f(x) = 4 - x$

74. $f(x) = 4x + 2$

75. $f(x) = x^2 - 9$

76. $f(x) = x^2 - 4x$

In Exercises 77 and 78, use a graphing utility to graph the function. State the domain and range of the function. Describe the pattern of the graph.

77. $s(x) = 2(\frac{1}{4}x - [\![\frac{1}{4}x]\!])$

78. $g(x) = 2(\frac{1}{4}x - [\![\frac{1}{4}x]\!])^2$

79. ***Geometry*** The perimeter of a rectangle is 100 meters.

(a) Show that the area of the rectangle is given by $A = x(50 - x)$, where x is its length.

(b) Use a graphing utility to graph the area function.

(c) Use a graphing utility to approximate the maximum area of the rectangle and the dimensions that yield the maximum area.

80. ***Cost, Revenue, and Profit*** The marketing department of a company estimates that the demand for a color scanner is $p = 100 - 0.0001x$, where p is the price per scanner and x is the number of scanners. The cost of producing x scanners is $C = 350{,}000 + 30x$ and the profit for producing and selling x scanners is

$$P = R - C = xp - C.$$

Use a graphing utility to graph the profit function and estimate the number of scanners that would produce a maximum profit.

81. ***Communications*** The cost of using a telephone calling card is \$1.05 for the first minute and \$0.38 for each additional minute or portion of a minute.

(a) A customer needs a model for the cost C of using the calling card for a call lasting t minutes. Which of the following is the appropriate model?

$$C_1(t) = 1.05 + 0.38[\![t - 1]\!]$$

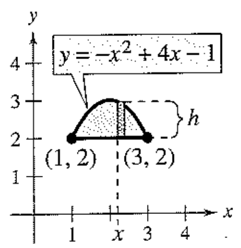
$$C_2(t) = 1.05 - 0.38[\![-(t - 1)]\!]$$

(b) Use a graphing utility to graph the appropriate model. Use the *value* feature or the *zoom* and *trace* features to estimate the cost of a call lasting 18 minutes and 45 seconds.

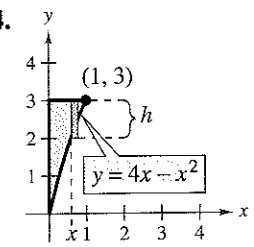
82. ***Delivery Charges*** The cost of sending an overnight package from New York to Atlanta is \$9.80 for a package weighing up to but not including 1 pound and \$2.50 for each additional pound or portion of a pound. Use the greatest integer function to create a model for the cost C of overnight delivery of a package weighing x pounds, where $x > 0$. Sketch the graph of the function.

In Exercises 83 and 84, write the height h of the rectangle as a function of x.

83.

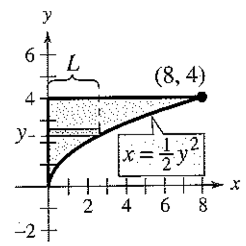


84.

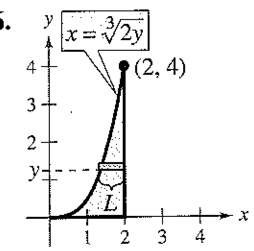


In Exercises 85 and 86, write the length L of the rectangle as a function of y.

85.



86.



87. ***Population*** During a seven-year period, the population P (in thousands) of North Dakota increased and then decreased according to the model

$$P = -0.76t^2 + 9.9t + 618, \quad 5 \leq t \leq 11$$

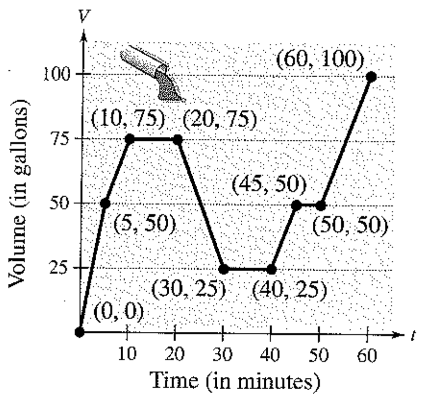
where t represents the year, with $t = 5$ corresponding to 1995. (Source: U.S. Census Bureau)

(a) Use a graphing utility to graph the model over the appropriate domain.

(b) Use the graph from part (a) to determine during which years the population was increasing. During which years was the population decreasing?

(c) Use the *zoom* and *trace* features or the *maximum* feature of a graphing utility to approximate the maximum population between 1995 and 2001.

88. ***Fluid Flow*** The intake pipe of a 100-gallon tank has a flow rate of 10 gallons per minute, and two drain pipes have a flow rate of 5 gallons per minute each. The graph shows the volume V of fluid in the tank as a function of time t. Determine in which pipes the fluid is flowing in specific subintervals of the one-hour interval of time shown on the graph. (There are many correct answers.)



Synthesis

True or False? **In Exercises 89 and 90, determine whether the statement is true or false. Justify your answer.**

89. A function with a square root cannot have a domain that is the set of all real numbers.

90. It is possible for an odd function to have the interval $[0, \infty)$ as its domain.

91. ***Proof*** Prove that a function of the following form is odd.

$$y = a_{2n+1}x^{2n+1} + a_{2n-1}x^{2n-1} + \cdot \cdot \cdot + a_3x^3 + a_1x$$

92. ***Proof*** Prove that a function of the following form is even.

$$y = a_{2n}x^{2n} + a_{2n-2}x^{2n-2} + \cdot \cdot \cdot + a_2x^2 + a_0$$

93. If f is an even function, determine if g is even, odd, or neither. Explain.

(a) $g(x) = -f(x)$ (b) $g(x) = f(-x)$

(c) $g(x) = f(x) - 2$ (d) $g(x) = -f(x - 2)$

94. ***Think About It*** Does the graph in Exercise 13 represent x as a function of y? Explain.

95. ***Think About It*** Does the graph in Exercise 14 represent x as a function of y? Explain.

96. ***Writing*** Write a short paragraph describing three different functions that represent the behaviors of quantities between 1990 and 2004. Describe one quantity that decreased during this time, one that increased, and one that was constant. Present your results graphically.

Review

In Exercises 97–100, identify the terms. Then identify the coefficients of the variable terms of the expression.

97. $-2x^2 + 8x$

98. $10 + 3x$

99. $\frac{x}{3} - 5x^2 + x^3$

100. $7x^4 + \sqrt{2}x^2$

In Exercises 101–104, find (a) the distance between the two points and (b) the midpoint of the line segment joining the points.

101. $(-2, 7), (6, 3)$

102. $(-5, 0), (3, 6)$

103. $\left(\frac{5}{2}, -1\right), \left(-\frac{3}{2}, 4\right)$

104. $\left(-6, \frac{2}{3}\right), \left(\frac{3}{4}, \frac{1}{6}\right)$

In Exercises 105–108, evaluate the function at each specified value of the independent variable and simplify.

105. $f(x) = 5x - 1$

(a) $f(6)$ (b) $f(-1)$ (c) $f(x - 3)$

106. $f(x) = -x^2 - x + 3$

(a) $f(4)$ (b) $f(-2)$ (c) $f(x - 2)$

107. $f(x) = x\sqrt{x - 3}$

(a) $f(3)$ (b) $f(12)$ (c) $f(6)$

108. $f(x) = -\frac{1}{2}x|x + 1|$

(a) $f(-4)$ (b) $f(10)$ (c) $f\left(-\frac{2}{3}\right)$

In Exercises 109 and 110, find the difference quotient and simplify your answer.

109. $f(x) = x^2 - 2x + 9$, $\frac{f(3 + h) - f(3)}{h}$, $h \neq 0$

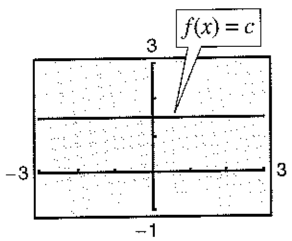
110. $f(x) = 5 + 6x - x^2$, $\frac{f(6 + h) - f(6)}{h}$, $h \neq 0$

1.4 Shifting, Reflecting, and Stretching Graphs

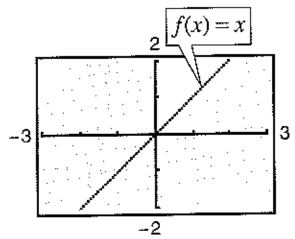
Summary of Graphs of Common Functions

One of the goals of this text is to enable you to build your intuition for the basic shapes of the graphs of different types of functions. For instance, from your study of lines in Section 1.1, you can determine the basic shape of the graph of the linear function $f(x) = mx + b$. Specifically, you know that the graph of this function is a line whose slope is m and whose y-intercept is $(0, b)$.

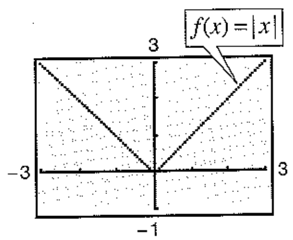
The six graphs shown in Figure 1.40 represent the most commonly used functions in algebra. Familiarity with the basic characteristics of these simple graphs will help you analyze the shapes of more complicated graphs.



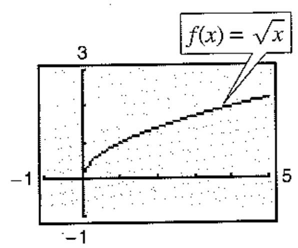
(a) Constant Function



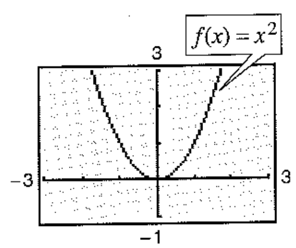
(b) Identity Function



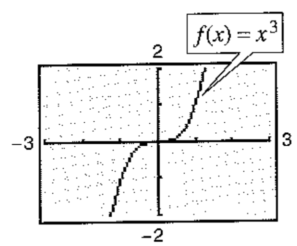
(c) Absolute Value Function



(d) Square Root Function



(e) Quadratic Function



(f) Cubic Function

Figure 1.40

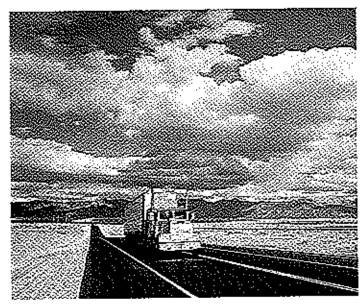
Throughout this section, you will discover how many complicated graphs are derived by shifting, stretching, shrinking, or reflecting the common graphs shown above. Shifts, stretches, shrinks, and reflections are called *transformations*. Many graphs of functions can be created from a combination of these transformations.

What you should learn

- Recognize graphs of common functions.
- Use vertical and horizontal shifts and reflections to graph functions.
- Use nonrigid transformations to graph functions.

Why you should learn it

Recognizing the graphs of common functions and knowing how to shift, reflect, and stretch graphs of functions can help you sketch a wide variety of simple functions by hand. This skill is useful in sketching graphs of functions that model real-life data. For example, in Exercise 67 on page 50, you are asked to sketch a function that models the amount of fuel used by trucks from 1980 through 2000.



Index Stock

Vertical and Horizontal Shifts

Many functions have graphs that are simple transformations of the common graphs summarized in Figure 1.40. For example, you can obtain the graph of

$$h(x) = x^2 + 2$$

by shifting the graph of $f(x) = x^2$ *upward* two units, as shown in Figure 1.41. In function notation, h and f are related as follows.

$$h(x) = x^2 + 2$$

$$= f(x) + 2 \quad \text{Upward shift of two units}$$

Similarly, you can obtain the graph of

$$g(x) = (x - 2)^2$$

by shifting the graph of $f(x) = x^2$ to the *right* two units, as shown in Figure 1.42. In this case, the functions g and f have the following relationship.

$$g(x) = (x - 2)^2$$

$$= f(x - 2) \quad \text{Right shift of two units}$$

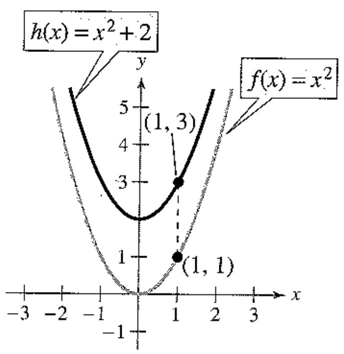


Figure 1.41 Vertical shift upward: two units

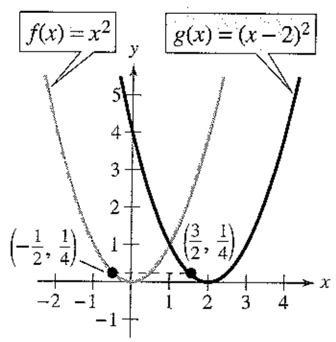


Figure 1.42 Horizontal shift to the right: two units

Exploration

Use a graphing utility to display (in the same viewing window) the graphs of $y = x^2 + c$, where $c = -2, 0, 2$, and 4. Use the result to describe the effect that c has on the graph.

Use a graphing utility to display (in the same viewing window) the graphs of $y = (x + c)^2$, where $c = -2, 0, 2$, and 4. Use the result to describe the effect that c has on the graph.

The following list summarizes horizontal and vertical shifts.

Vertical and Horizontal Shifts

Let c be a positive real number. **Vertical and horizontal shifts** in the graph of $y = f(x)$ are represented as follows.

1. Vertical shift c units *upward:* $\quad h(x) = f(x) + c$
2. Vertical shift c units *downward:* $\quad h(x) = f(x) - c$
3. Horizontal shift c units to the *right:* $\quad h(x) = f(x - c)$
4. Horizontal shift c units to the *left:* $\quad h(x) = f(x + c)$

In items 3 and 4, be sure you see that $h(x) = f(x - c)$ corresponds to a *right* shift and $h(x) = f(x + c)$ corresponds to a *left* shift for $c > 0$.

Example 1 Shifts in the Graph of a Function

Compare the graph of each function with the graph of $f(x) = x^3$.

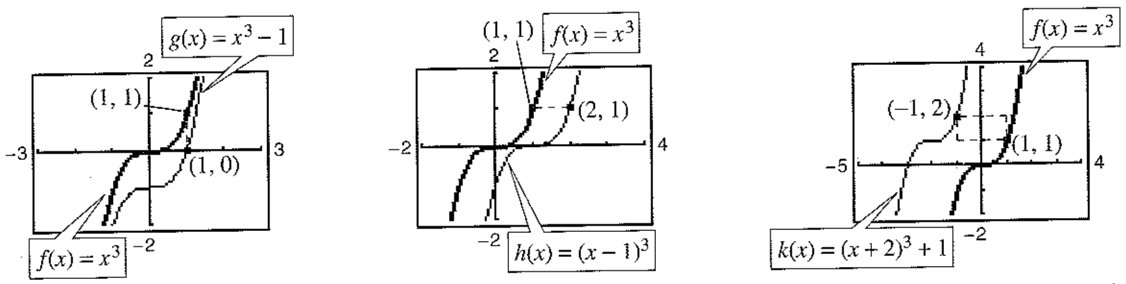
a. $g(x) = x^3 - 1$ **b.** $h(x) = (x - 1)^3$ **c.** $k(x) = (x + 2)^3 + 1$

Solution

a. Graph $f(x) = x^3$ and $g(x) = x^3 - 1$ [see Figure 1.43(a)]. You can obtain the graph of g by shifting the graph of f one unit downward.

b. Graph $f(x) = x^3$ and $h(x) = (x - 1)^3$ [see Figure 1.43(b)]. You can obtain the graph of h by shifting the graph of f one unit to the right.

c. Graph $f(x) = x^3$ and $k(x) = (x + 2)^3 + 1$ [see Figure 1.43(c)]. You can obtain the graph of k by shifting the graph of f two units to the left and then one unit upward.



(a) Vertical shift: one unit downward (b) Horizontal shift: one unit right (c) Two units left and one unit upward

Figure 1.43

✓ *Checkpoint* Now try Exercise 3.

Example 2 Finding Equations from Graphs

The graph of $f(x) = x^2$ is shown in Figure 1.44. Each of the graphs in Figure 1.45 is a transformation of the graph of f. Find an equation for each function.

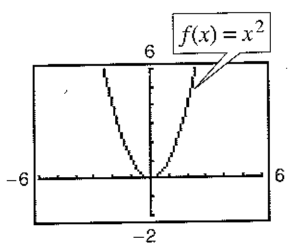
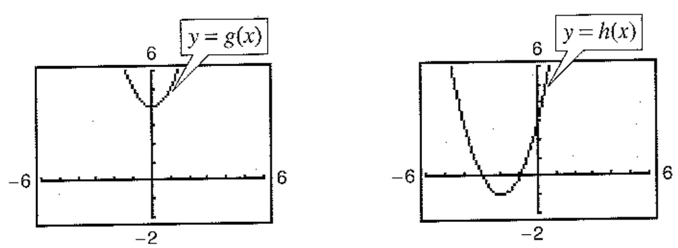


Figure 1.44



(a) (b)

Figure 1.45

Solution

a. The graph of g is a vertical shift of four units upward of the graph of $f(x) = x^2$. So, the equation for g is $g(x) = x^2 + 4$.

b. The graph of h is a horizontal shift of two units to the left, and a vertical shift of one unit downward, of the graph of $f(x) = x^2$. So, the equation for h is $h(x) = (x + 2)^2 - 1$.

✓ *Checkpoint* Now try Exercise 21.

Reflecting Graphs

The second common type of transformation is called a **reflection**. For instance, if you consider the x-axis to be a mirror, the graph of $h(x) = -x^2$ is the mirror image (or reflection) of the graph of $f(x) = x^2$ (see Figure 1.46).

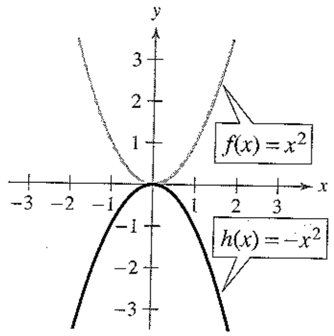


Figure 1.46

Exploration

Compare the graph of each function with the graph of $f(x) = x^2$ by using a graphing utility to graph the function and f in the same viewing window. Describe the transformation.

a. $g(x) = -x^2$

b. $h(x) = (-x)^2$

Reflections in the Coordinate Axes

Reflections in the coordinate axes of the graph of $y = f(x)$ are represented as follows.

1. Reflection in the x-axis: $h(x) = -f(x)$
2. Reflection in the y-axis: $h(x) = f(-x)$

Example 3 Finding Equations from Graphs

The graph of $f(x) = x^4$ is shown in Figure 1.47. Each of the graphs in Figure 1.48 is a transformation of the graph of f. Find an equation for each function.

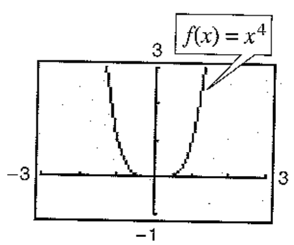
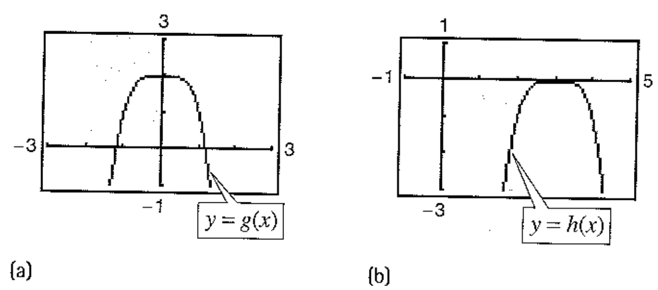


Figure 1.47



(a) (b)

Figure 1.48

Solution

a. The graph of g is a reflection in the x-axis *followed by* an upward shift of two units of the graph of $f(x) = x^4$. So, the equation for g is $g(x) = -x^4 + 2$.

b. The graph of h is a horizontal shift of three units to the right *followed by* a reflection in the x-axis of the graph of $f(x) = x^4$. So, the equation for h is $h(x) = -(x - 3)^4$.

Checkpoint Now try Exercise 25.

Example 4 Reflections and Shifts

Compare the graph of each function with the graph of $f(x) = \sqrt{x}$.

a. $g(x) = -\sqrt{x}$ **b.** $h(x) = \sqrt{-x}$ **c.** $k(x) = -\sqrt{x+2}$

Algebraic Solution

a. Relative to the graph of $f(x) = \sqrt{x}$, the graph of g is a reflection in the x-axis because

$$g(x) = -\sqrt{x}$$
$$= -f(x).$$

b. The graph of h is a reflection of the graph of $f(x) = \sqrt{x}$ in the y-axis because

$$h(x) = \sqrt{-x}$$
$$= f(-x).$$

c. From the equation

$$k(x) = -\sqrt{x+2}$$
$$= -f(x+2)$$

you can conclude that the graph of k is a left shift of two units, followed by a reflection in the x-axis, of the graph of $f(x) = \sqrt{x}$.

Checkpoint Now try Exercise 27.

Graphical Solution

a. Use a graphing utility to graph f and g in the same viewing window. From the graph in Figure 1.49, you can see that the graph of g is a reflection of the graph of f in the x-axis.

b. Use a graphing utility to graph f and h in the same viewing window. From the graph in Figure 1.50, you can see that the graph of h is a reflection of the graph of f in the y-axis.

c. Use a graphing utility to graph f and k in the same viewing window. From the graph in Figure 1.51, you can see that the graph of k is a left shift of two units of the graph of f, followed by a reflection in the x-axis.

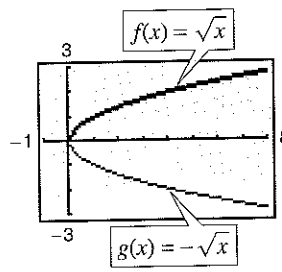


Figure 1.49

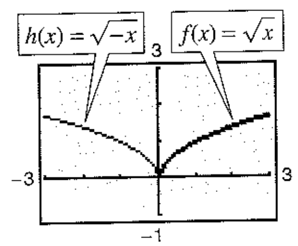


Figure 1.50

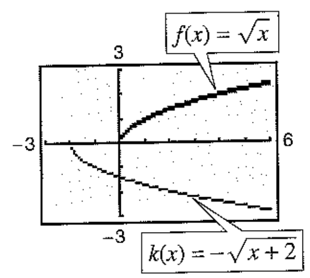


Figure 1.51

When graphing functions involving square roots, remember that the domain must be restricted to exclude negative numbers inside the radical. For instance, here are the domains of the functions in Example 4.

Domain of $g(x) = -\sqrt{x}$: $x \geq 0$

Domain of $h(x) = \sqrt{-x}$: $x \leq 0$

Domain of $k(x) = -\sqrt{x+2}$: $x \geq -2$

Nonrigid Transformations

Horizontal shifts, vertical shifts, and reflections are called **rigid transformations** because the basic shape of the graph is unchanged. These transformations change only the *position* of the graph in the coordinate plane. **Nonrigid transformations** are those that cause a *distortion*—a change in the shape of the original graph. For instance, a nonrigid transformation of the graph of $y = f(x)$ is represented by $y = cf(x)$ (each y-value is multiplied by c), where the transformation is a **vertical stretch** if $c > 1$ and a **vertical shrink** if $0 < c < 1$. Another nonrigid transformation of the graph of $y = f(x)$ is represented by $h(x) = f(cx)$ (each x-value is multiplied by $1/c$), where the transformation is a **horizontal shrink** if $c > 1$ and a **horizontal stretch** if $0 < c < 1$.

Example 5 Nonrigid Transformations

Compare the graph of each function with the graph of $f(x) = |x|$.

a. $h(x) = 3|x|$ **b.** $g(x) = \frac{1}{3}|x|$

Solution

a. Relative to the graph of $f(x) = |x|$, the graph of

$$h(x) = 3|x| = 3f(x)$$

is a vertical stretch (each y-value is multiplied by 3) of the graph of f. (See Figure 1.52.)

b. Similarly, the graph of

$$g(x) = \frac{1}{3}|x| = \frac{1}{3}f(x)$$

is a vertical shrink $\left(\text{each } y\text{-value is multiplied by } \frac{1}{3}\right)$ of the graph of f. (See Figure 1.53.)

Checkpoint Now try Exercise 37.

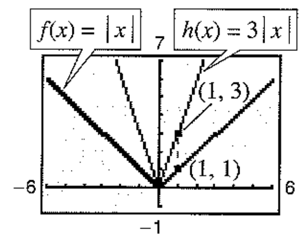


Figure 1.52

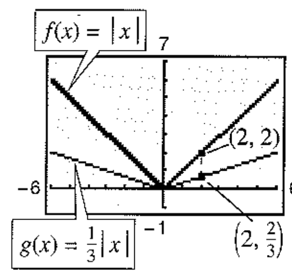


Figure 1.53

Example 6 Nonrigid Transformations

Compare the graph of $h(x) = f\left(\frac{1}{2}x\right)$ with the graph of $f(x) = 2 - x^3$.

Solution

Relative to the graph of $f(x) = 2 - x^3$, the graph of

$$h(x) = f\left(\tfrac{1}{2}x\right) = 2 - \left(\tfrac{1}{2}x\right)^3 = 2 - \tfrac{1}{8}x^3$$

is a horizontal stretch (each x-value is multiplied by 2) of the graph of f. (See Figure 1.54.)

Checkpoint Now try Exercise 43.

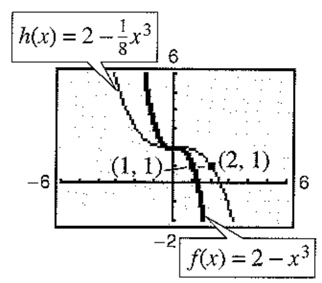


Figure 1.54

1.4 Exercises

Vocabulary Check

In Exercises 1–5, fill in the blanks.

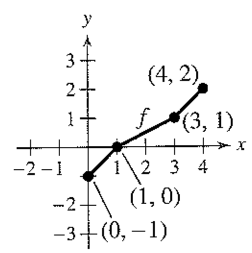
1. The graph of a ________ is U-shaped.
2. The graph of an ________ is V-shaped.
3. Horizontal shifts, vertical shifts, and reflections are called ________ .
4. A reflection in the x-axis of $y = f(x)$ is represented by $h(x) =$ ________ , while a reflection in the y-axis of $y = f(x)$ is represented by $h(x) =$ ________ .
5. A nonrigid transformation of $y = f(x)$ represented by $cf(x)$ is a vertical stretch if ________ and a vertical shrink if ________ .
6. Match the rigid transformation of $y = f(x)$ with the correct representation, where $c > 0$.

(a) $h(x) = f(x) + c$	(i) horizontal shift c units to the left
(b) $h(x) = f(x) - c$	(ii) vertical shift c units upward
(c) $h(x) = f(x - c)$	(iii) horizontal shift c units to the right
(d) $h(x) = f(x + c)$	(iv) vertical shift c units downward

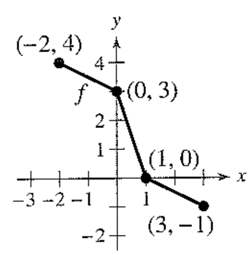
In Exercises 1–12, sketch the graphs of the three functions by hand on the same rectangular coordinate system. Verify your result with a graphing utility.

1. $f(x) = x$
   $g(x) = x - 4$
   $h(x) = 3x$
2. $f(x) = \frac{1}{2}x$
   $g(x) = \frac{1}{2}x + 2$
   $h(x) = \frac{1}{2}(x - 2)$
3. $f(x) = x^2$
   $g(x) = x^2 + 2$
   $h(x) = (x - 2)^2$
4. $f(x) = x^2$
   $g(x) = x^2 - 4$
   $h(x) = (x + 2)^2 + 1$
5. $f(x) = -x^2$
   $g(x) = -x^2 + 1$
   $h(x) = -(x - 2)^2$
6. $f(x) = (x - 2)^2$
   $g(x) = (x - 2)^2 + 2$
   $h(x) = -(x - 2)^2 + 4$
7. $f(x) = x^2$
   $g(x) = \frac{1}{2}x^2$
   $h(x) = (2x)^2$
8. $f(x) = x^2$
   $g(x) = \frac{1}{4}x^2 + 2$
   $h(x) = -\frac{1}{4}x^2$
9. $f(x) = |x|$
   $g(x) = |x| - 1$
   $h(x) = |x - 3|$
10. $f(x) = |x|$
    $g(x) = |2x|$
    $h(x) = -2|x + 2| - 1$
11. $f(x) = \sqrt{x}$
    $g(x) = \sqrt{x + 1}$
    $h(x) = \sqrt{x - 2} + 1$
12. $f(x) = \sqrt{x}$
    $g(x) = \frac{1}{2}\sqrt{x}$
    $h(x) = -\frac{1}{2}\sqrt{x + 4}$

13. Use the graph of f to sketch each graph. To print an enlarged copy of the graph, go to the website *www.mathgraphs.com.*
    (a) $y = f(x) + 2$
    (b) $y = -f(x)$
    (c) $y = f(x - 2)$
    (d) $y = f(x + 3)$
    (e) $y = 2f(x)$
    (f) $y = f(-x)$
    (g) $y = f(\frac{1}{2}x)$

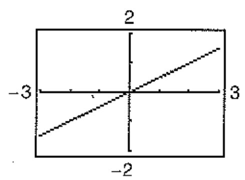


14. Use the graph of f to sketch each graph. To print an enlarged copy of the graph, go to the website *www.mathgraphs.com.*
    (a) $y = f(x) - 1$
    (b) $y = f(x + 1)$
    (c) $y = f(x - 1)$
    (d) $y = -f(x - 2)$
    (e) $y = f(-x)$
    (f) $y = \frac{1}{2}f(x)$
    (g) $y = f(2x)$



In Exercises 15–26, identify the common function and describe the transformation shown in the graph. Write an equation for the graphed function.

15.



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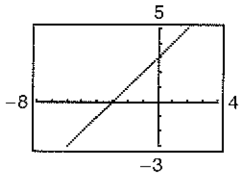

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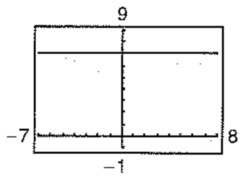

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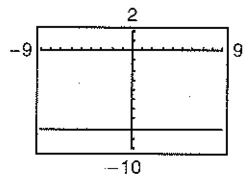

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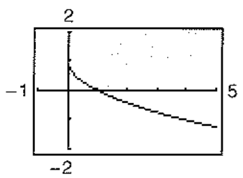

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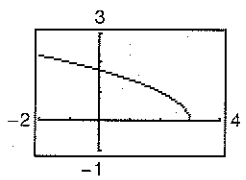

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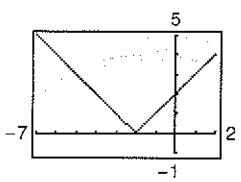

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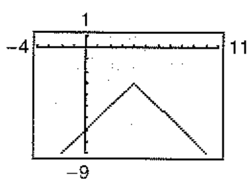

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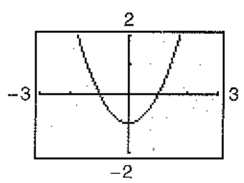

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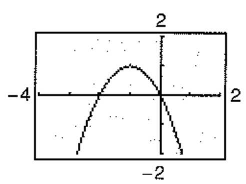

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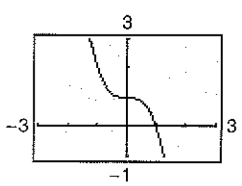

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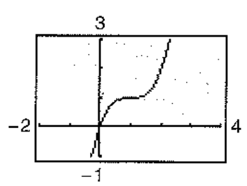


In Exercises 27–32, compare the graph of the function with the graph of $f(x) = \sqrt{x}$.

27. $y = -\sqrt{x} - 1$ **28.** $y = \sqrt{x} + 2$

29. $y = \sqrt{x - 2}$ **30.** $y = \sqrt{x + 4}$

31. $y = \sqrt{2x}$ **32.** $y = \sqrt{-x + 3}$

In Exercises 33–38, compare the graph of the function with the graph of $f(x) = |x|$.

33. $y = |x + 5|$ **34.** $y = |x| - 3$

35. $y = -|x|$ **36.** $y = |-x|$

37. $y = 4|x|$ **38.** $y = |\frac{1}{2}x|$

In Exercises 39–44, compare the graph of the function with the graph of $f(x) = x^3$.

39. $g(x) = 4 - x^3$ **40.** $g(x) = -(x - 1)^3$

41. $h(x) = \frac{1}{4}(x + 2)^3$ **42.** $h(x) = -2(x - 1)^3 + 3$

43. $p(x) = \left(\frac{1}{3}x\right)^3 + 2$ **44.** $p(x) = [3(x - 2)]^3$

In Exercises 45–48, use a graphing utility to graph the three functions in the same viewing window. Describe the graphs of g and h relative to the graph of f.

45. $f(x) = x^3 - 3x^2$
$g(x) = f(x + 2)$
$h(x) = \frac{1}{2}f(x)$

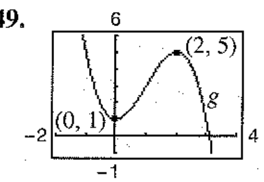
46. $f(x) = x^3 - 3x^2 + 2$
$g(x) = f(x - 1)$
$h(x) = f(3x)$

47. $f(x) = x^3 - 3x^2$
$g(x) = -\frac{1}{3}f(x)$
$h(x) = f(-x)$

48. $f(x) = x^3 - 3x^2 + 2$
$g(x) = -f(x)$
$h(x) = f(2x)$

In Exercises 49 and 50, use the graph of $f(x) = x^3 - 3x^2$ (see Exercise 45) to write a formula for the function g shown in the graph.

49.



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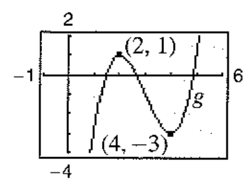


In Exercises 51–64, g is related to one of the six common functions on page 42. (a) Identify the common function f. (b) Describe the sequence of transformations from f to g. (c) Sketch the graph of g by hand. (d) Use function notation to write g in terms of the common function f.

51. $g(x) = 2 - (x + 5)^2$

52. $g(x) = -(x + 10)^2 + 5$

53. $g(x) = 3 + 2(x - 4)^2$

54. $g(x) = -\frac{1}{4}(x + 2)^2 - 2$

55. $g(x) = 3(x - 2)^3$ **56.** $g(x) = -\frac{1}{2}(x + 1)^3$

57. $g(x) = (x - 1)^3 + 2$

58. $g(x) = -(x + 3)^3 - 10$

59. $g(x) = |x + 4| + 8$

60. $g(x) = |x + 3| + 9$

61. $g(x) = -2|x - 1| - 4$

62. $g(x) = \frac{1}{2}|x - 2| - 3$

63. $g(x) = -\frac{1}{2}\sqrt{x + 3} - 1$

64. $g(x) = -\sqrt{x + 1} - 6$

65. ***Profit*** The profit P per week on a case of soda pop is given by the model

$$P(x) = 80 + 20x - 0.5x^2, \qquad 0 \le x \le 20$$

where x is the amount spent on advertising. In this model, x and P are both measured in hundreds of dollars.

(a) Use a graphing utility to graph the profit function.

(b) The business estimates that taxes and operating costs will increase by an average of \$2500 per week during the next year. Rewrite the profit function to reflect this expected decrease in profits. Describe the transformation applied to the graph of the function.

(c) Rewrite the profit function so that x measures advertising expenditures in dollars. [Find $P\left(\frac{x}{100}\right)$.] Describe the transformation applied to the graph of the profit function.

66. ***Automobile Aerodynamics*** The number of horsepower H required to overcome wind drag on an automobile is approximated by the model

$$H(x) = 0.002x^2 + 0.005x - 0.029, \qquad 10 \le x \le 100$$

where x is the speed of the car in miles per hour.

(a) Use a graphing utility to graph the function.

(b) Rewrite the function so that x represents the speed in kilometers per hour. [Find $H(x/1.6)$.] Describe the transformation applied to the graph of the function.

67. ***Fuel Use*** The amount of fuel F (in billions of gallons) used by trucks from 1980 through 2000 can be approximated by the function $F(t) = 0.036t^2 + 20.1$, where $t = 0$ represents 1980. (Source: U.S. Federal Highway Administration)

(a) Describe the transformation of the common function $f(t) = t^2$. Then sketch the graph over the interval $0 \le t \le 20$.

(b) Rewrite the function so that $t = 0$ represents 1990. Explain how you got your answer.

68. ***Finance*** The amount M (in billions of dollars) of mortgage debt outstanding in the United States from 1990 through 2001 can be approximated by the function $M(t) = 29.9t^2 + 3892$, where $t = 0$ represents 1990. (Source: Board of Governors of the Federal Reserve System)

(a) Describe the transformation of the common function $f(t) = t^2$. Then sketch the graph over the interval $0 \le t \le 11$.

(b) Rewrite the function so that $t = 0$ represents 2000. Explain how you got your answer.

Synthesis

True or False? **In Exercises 69 and 70, determine whether the statement is true or false. Justify your answer.**

69. The graphs of $f(x) = |x| - 5$ and $g(x) = |-x| - 5$ are identical.

70. Relative to the graph of $f(x) = \sqrt{x}$, the graph of the function $h(x) = -\sqrt{x + 9} - 13$ is shifted 9 units to the left and 13 units downward, then reflected in the x-axis.

71. ***Exploration*** Use a graphing utility to graph each function. Describe any similarities and differences you observe among the graphs.

(a) $y = x$ (b) $y = x^2$ (c) $y = x^3$

(d) $y = x^4$ (e) $y = x^5$ (f) $y = x^6$

72. ***Conjecture*** Use the results of Exercise 71.

(a) Make a conjecture about the shapes of the graphs of $y = x^7$ and $y = x^8$. Use a graphing utility to verify your conjecture.

(b) Sketch the graphs of $y = (x - 3)^3$ and $y = (x + 1)^2$ by hand. Use a graphing utility to verify your graphs.

Review

In Exercises 73 and 74, determine whether the lines L_1 and L_2 passing through the pairs of points are parallel, perpendicular, or neither.

73. L_1: $(-2, -2), (2, 10)$
L_2: $(-1, 3), (3, 9)$

74. L_1: $(-1, -7), (4, 3)$
L_2: $(1, 5), (-2, -7)$

In Exercises 75–78, find the domain of the function.

75. $f(x) = \dfrac{4}{9 - x}$

76. $f(x) = \dfrac{\sqrt{x - 5}}{x - 7}$

77. $f(x) = \sqrt{100 - x^2}$

78. $f(x) = \sqrt[3]{16 - x^2}$

1.5 Combinations of Functions

Arithmetic Combinations of Functions

Just as two real numbers can be combined by the operations of addition, subtraction, multiplication, and division to form other real numbers, two *functions* can be combined to create new functions. If $f(x) = 2x - 3$ and $g(x) = x^2 - 1$, you can form the sum, difference, product, and quotient of f and g as follows.

$$f(x) + g(x) = (2x - 3) + (x^2 - 1)$$
$$= x^2 + 2x - 4 \quad \text{Sum}$$
$$f(x) - g(x) = (2x - 3) - (x^2 - 1)$$
$$= -x^2 + 2x - 2 \quad \text{Difference}$$
$$f(x) \cdot g(x) = (2x - 3)(x^2 - 1)$$
$$= 2x^3 - 3x^2 - 2x + 3 \quad \text{Product}$$
$$\frac{f(x)}{g(x)} = \frac{2x - 3}{x^2 - 1}, \quad x \neq \pm 1 \quad \text{Quotient}$$

The domain of an **arithmetic combination** of functions f and g consists of all real numbers that are common to the domains of f and g. In the case of the quotient $f(x)/g(x)$, there is the further restriction that $g(x) \neq 0$.

Sum, Difference, Product, and Quotient of Functions

Let f and g be two functions with overlapping domains. Then, for all x common to both domains, the sum, difference, product, and quotient of f and g are defined as follows.

1. Sum: $(f + g)(x) = f(x) + g(x)$
2. Difference: $(f - g)(x) = f(x) - g(x)$
3. Product: $(fg)(x) = f(x) \cdot g(x)$
4. Quotient: $\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)}, \quad g(x) \neq 0$

Example 1 Finding the Sum of Two Functions

Given $f(x) = 2x + 1$ and $g(x) = x^2 + 2x - 1$, find $(f + g)(x)$. Then evaluate the sum when $x = 2$.

Solution

$$(f + g)(x) = f(x) + g(x) = (2x + 1) + (x^2 + 2x - 1) = x^2 + 4x$$

When $x = 2$, the value of this sum is $(f + g)(2) = 2^2 + 4(2) = 12$.

✓ ***Checkpoint*** Now try Exercise 13.

What you should learn

- Add, subtract, multiply, and divide functions.
- Find compositions of one function with another function.
- Use combinations of functions to model and solve real-life problems.

Why you should learn it

Combining functions can sometimes help you better understand the big picture. For instance, Exercises 75 and 60 on page 60 illustrate how to use combinations of functions to analyze U.S. health expenditures.

SuperStock

Example 2 Finding the Difference of Two Functions

Given $f(x) = 2x + 1$ and $g(x) = x^2 + 2x - 1$, find $(f - g)(x)$. Then evaluate the difference when $x = 2$.

Algebraic Solution

The difference of the functions f and g is

$$\begin{aligned}(f - g)(x) &= f(x) - g(x)\\ &= (2x + 1) - (x^2 + 2x - 1)\\ &= -x^2 + 2.\end{aligned}$$

When $x = 2$, the value of this difference is

$$\begin{aligned}(f - g)(2) &= -(2)^2 + 2\\ &= -2.\end{aligned}$$

Note that $(f - g)(2)$ can also be evaluated as follows.

$$\begin{aligned}(f - g)(2) &= f(2) - g(2)\\ &= [2(2) + 1] - [2^2 + 2(2) - 1]\\ &= 5 - 7\\ &= -2\end{aligned}$$

✓ *Checkpoint* Now try Exercise 15.

Graphical Solution

You can use a graphing utility to graph the difference of two functions. Enter the functions as follows (see Figure 1.55).

$$y_1 = 2x + 1$$

$$y_2 = x^2 + 2x - 1$$

$$y_3 = y_1 - y_2$$

Graph y_3 as shown in Figure 1.56. Then use the *value* feature or the *zoom* and *trace* features to estimate that the value of the difference when $x = 2$ is -2.

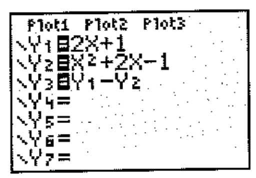


Figure 1.55

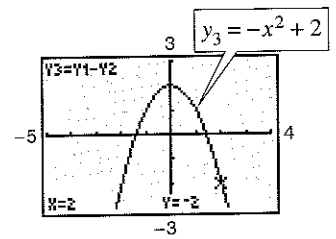


Figure 1.56

In Examples 1 and 2, both f and g have domains that consist of all real numbers. So, the domain of both $(f + g)$ and $(f - g)$ is also the set of all real numbers. Remember that any restrictions on the domains of f or g must be considered when forming the sum, difference, product, or quotient of f and g. For instance, the domain of $f(x) = 1/x$ is all $x \neq 0$, and the domain of $g(x) = \sqrt{x}$ is $[0, \infty)$. This implies that the domain of $(f + g)$ is $(0, \infty)$.

Example 3 Finding the Product of Two Functions

Given $f(x) = x^2$ and $g(x) = x - 3$, find $(fg)(x)$. Then evaluate the product when $x = 4$.

Solution

$$\begin{aligned}(fg)(x) &= f(x)g(x)\\ &= (x^2)(x - 3)\\ &= x^3 - 3x^2\end{aligned}$$

When $x = 4$, the value of this product is

$$(fg)(4) = 4^3 - 3(4)^2 = 16.$$

✓ *Checkpoint* Now try Exercise 17.

Example 4 Finding the Quotient of Two Functions

Find $(f/g)(x)$ and $(g/f)(x)$ for the functions given by $f(x) = \sqrt{x}$ and $g(x) = \sqrt{4 - x^2}$. Then find the domains of f/g and g/f.

Solution

The quotient of f and g is

$$\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)} = \frac{\sqrt{x}}{\sqrt{4 - x^2}},$$

and the quotient of g and f is

$$\left(\frac{g}{f}\right)(x) = \frac{g(x)}{f(x)} = \frac{\sqrt{4 - x^2}}{\sqrt{x}}.$$

The domain of f is $[0, \infty)$ and the domain of g is $[-2, 2]$. The intersection of these domains is $[0, 2]$. So, the domains for f/g and g/f are as follows.

Domain of $(f/g): [0, 2)$ Domain of $(g/f): (0, 2]$

Checkpoint Now try Exercise 19.

TECHNOLOGY TIP You can confirm the domain of f/g in Example 4 with your graphing utility by entering the three functions $y_1 = \sqrt{x}$, $y_2 = \sqrt{4 - x^2}$, and $y_3 = y_1/y_2$, and graphing y_3 as shown in Figure 1.57. Use the *trace* feature to determine that the x-coordinates of points on the graph extend from 0 to 2 but do not include 2. So, you can estimate the domain of f/g to be $[0, 2)$. You can confirm the domain of g/f in Example 4 by entering $y_4 = y_2/y_1$ and graphing y_4 as shown in Figure 1.58. Use the *trace* feature to determine that the x-coordinates of points on the graph extend from 0 to 2 but do not include 0. So, you can estimate the domain of g/f to be $(0, 2]$.

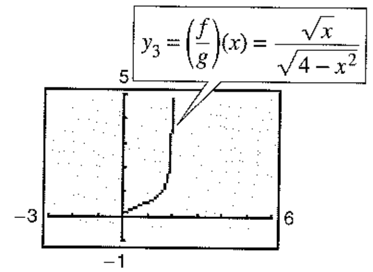


Figure 1.57

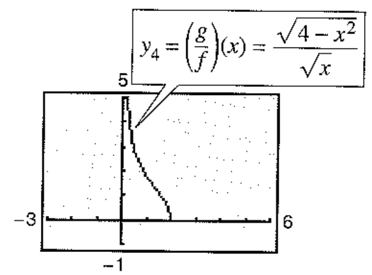


Figure 1.58

Compositions of Functions

Another way of combining two functions is to form the **composition** of one with the other. For instance, if $f(x) = x^2$ and $g(x) = x + 1$, the composition of f with g is

$$f(g(x)) = f(x + 1) = (x + 1)^2.$$

This composition is denoted as $f \circ g$ and read as "f of g."

Definition of Composition of Two Functions

The **composition** of the function f with the function g is

$$(f \circ g)(x) = f(g(x)).$$

The domain of $f \circ g$ is the set of all x in the domain of g such that $g(x)$ is in the domain of f. (See Figure 1.59.)

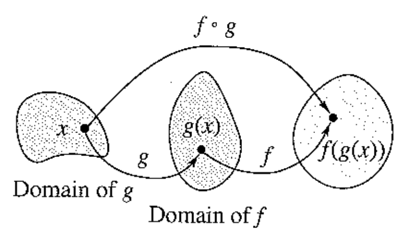


Figure 1.59

Example 5 Forming the Composition of f with g

Find $(f \circ g)(x)$ for $f(x) = \sqrt{x}$, $x \geq 0$, and $g(x) = x - 1$, $x \geq 1$. If possible, find $(f \circ g)(2)$ and $(f \circ g)(0)$.

Solution

$$(f \circ g)(x) = f(g(x)) \qquad \text{Definition of } f \circ g$$
$$= f(x - 1) \qquad \text{Definition of } g(x)$$
$$= \sqrt{x - 1}, \quad x \geq 1 \qquad \text{Definition of } f(x)$$

The domain of $f \circ g$ is $[1, \infty)$. So, $(f \circ g)(2) = \sqrt{2 - 1} = 1$ is defined, but $(f \circ g)(0)$ is not defined because 0 is not in the domain of $f \circ g$.

✓ ***Checkpoint*** Now try Exercise 35.

The composition of f with g is generally not the same as the composition of g with f. This is illustrated in Example 6.

Exploration

Let $f(x) = x + 2$ and $g(x) = 4 - x^2$. Are the compositions $f \circ g$ and $g \circ f$ equal? You can use your graphing utility to answer this question by entering and graphing the following functions.

$$y_1 = (4 - x^2) + 2$$
$$y_2 = 4 - (x + 2)^2$$

What do you observe? Which function represents $f \circ g$ and which represents $g \circ f$?

Example 6 Compositions of Functions

Given $f(x) = x + 2$ and $g(x) = 4 - x^2$, evaluate (a) $(f \circ g)(x)$ and (b) $(g \circ f)(x)$ when $x = 0, 1, 2$, and 3.

Algebraic Solution

a. $(f \circ g)(x) = f(g(x))$ — Definition of $f \circ g$

$= f(4 - x^2)$ — Definition of $g(x)$

$= (4 - x^2) + 2$ — Definition of $f(x)$

$= -x^2 + 6$

$(f \circ g)(0) = -0^2 + 6 = 6$

$(f \circ g)(1) = -1^2 + 6 = 5$

$(f \circ g)(2) = -2^2 + 6 = 2$

$(f \circ g)(3) = -3^2 + 6 = -3$

b. $(g \circ f)(x) = g(f(x))$ — Definition of $g \circ f$

$= g(x + 2)$ — Definition of $f(x)$

$= 4 - (x + 2)^2$ — Definition of $g(x)$

$= 4 - (x^2 + 4x + 4)$

$= -x^2 - 4x$

$(g \circ f)(0) = -0^2 - 4(0) = 0$

$(g \circ f)(1) = -1^2 - 4(1) = -5$

$(g \circ f)(2) = -2^2 - 4(2) = -12$

$(g \circ f)(3) = -3^2 - 4(3) = -21$

Note that $f \circ g \neq g \circ f$.

✓ ***Checkpoint*** Now try Exercise 37.

Numerical Solution

a. You can use the *table* feature of a graphing utility to evaluate $f \circ g$ when $x = 0, 1, 2$, and 3. Enter $y_1 = g(x)$ and $y_2 = f(g(x))$ in the equation editor (see Figure 1.60). Then set the table to *ask* mode to find the desired function values (see Figure 1.61). Finally, display the table, as shown in Figure 1.62.

b. You can evaluate $g \circ f$ when $x = 0, 1, 2$, and 3 by using a procedure similar to that of part (a). You should obtain the table shown in Figure 1.63.

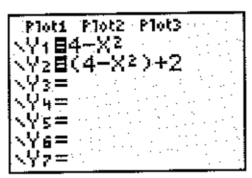


Figure 1.60

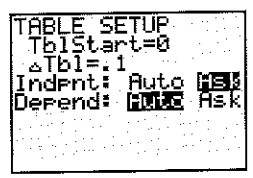


Figure 1.61

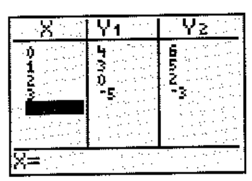


Figure 1.62

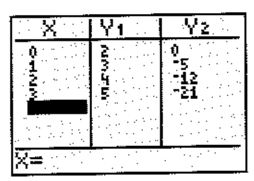


Figure 1.63

From the tables you can see that $f \circ g \neq g \circ f$.

To determine the domain of a composite function $f \circ g$, you need to restrict the outputs of g so that they are in the domain of f. For instance, to find the domain of $f \circ g$ given that $f(x) = 1/x$ and $g(x) = x + 1$, consider the outputs of g. These can be any real number. However, the domain of f is restricted to all real numbers except 0. So, the outputs of g must be restricted to all real numbers except 0. This means that $g(x) \neq 0$, or $x \neq -1$. So, the domain of $f \circ g$ is all real numbers except $x = -1$.

Example 7 Finding the Domain of a Composite Function

Find the domain of the composition $(f \circ g)(x)$ for the functions given by

$$f(x) = x^2 - 9 \qquad \text{and} \qquad g(x) = \sqrt{9 - x^2}.$$

Algebraic Solution

The composition of the functions is as follows.

$$\begin{aligned}(f \circ g)(x) &= f(g(x)) \\ &= f\left(\sqrt{9 - x^2}\right) \\ &= \left(\sqrt{9 - x^2}\right)^2 - 9 \\ &= 9 - x^2 - 9 \\ &= -x^2\end{aligned}$$

From this, it might appear that the domain of the composition is the set of all real numbers. This, however, is not true. Because the domain of f is the set of all real numbers and the domain of g is $[-3, 3]$, the domain of $(f \circ g)$ is $[-3, 3]$.

✓ *Checkpoint* Now try Exercise 39.

Graphical Solution

You can use a graphing utility to graph the composition of the functions $(f \circ g)(x)$ as $y = \left(\sqrt{9 - x^2}\right)^2 - 9$. Enter the functions as follows.

$$y_1 = \sqrt{9 - x^2} \qquad y_2 = y_1^2 - 9$$

Graph y_2 as shown in Figure 1.64. Use the *trace* feature to determine that the x-coordinates of points on the graph extend from -3 to 3. So, you can graphically estimate the domain of $(f \circ g)(x)$ to be $[-3, 3]$.

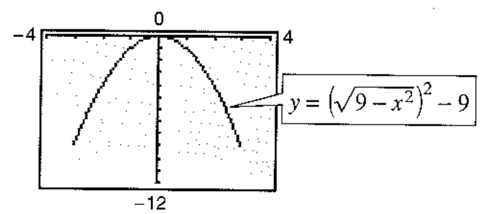


Figure 1.64

Example 8 A Case in Which $f \circ g = g \circ f$

Given $f(x) = 2x + 3$ and $g(x) = \frac{1}{2}(x - 3)$, find each composition.

a. $(f \circ g)(x)$ **b.** $(g \circ f)(x)$

Solution

a.
$$\begin{aligned}(f \circ g)(x) &= f(g(x)) \\ &= f\left(\frac{1}{2}(x - 3)\right) \\ &= 2\left[\frac{1}{2}(x - 3)\right] + 3 \\ &= x - 3 + 3 = x\end{aligned}$$

b.
$$\begin{aligned}(g \circ f)(x) &= g(f(x)) \\ &= g(2x + 3) \\ &= \frac{1}{2}\left[(2x + 3) - 3\right] \\ &= \frac{1}{2}(2x) = x\end{aligned}$$

✓ *Checkpoint* Now try Exercise 43.

STUDY TIP

In Example 8, note that the two composite functions $f \circ g$ and $g \circ f$ are equal, and both represent the identity function. That is, $(f \circ g)(x) = x$ and $(g \circ f)(x) = x$. You will study this special case in the next section.

In Examples 5, 6, 7, and 8 you formed the composition of two given functions. In calculus, it is also important to be able to identify two functions that make up a given composite function. Basically, to "decompose" a composite function, look for an "inner" and an "outer" function.

Example 9 Identifying a Composite Function

Write the function $h(x) = (3x - 5)^3$ as a composition of two functions.

Solution

One way to write h as a composition of two functions is to take the inner function to be $g(x) = 3x - 5$ and the outer function to be $f(x) = x^3$. Then you can write

$$\begin{aligned} h(x) &= (3x - 5)^3 \\ &= f(3x - 5) \\ &= f(g(x)). \end{aligned}$$

✓ ***Checkpoint*** Now try Exercise 55.

Example 10 Identifying a Composite Function

Write the function

$$h(x) = \frac{1}{(x - 2)^2}$$

as a composition of two functions.

Solution

One way to write h as a composition of two functions is to take the inner function to be $g(x) = x - 2$ and the outer function to be

$$\begin{aligned} f(x) &= \frac{1}{x^2} \\ &= x^{-2}. \end{aligned}$$

Then you can write

$$\begin{aligned} h(x) &= \frac{1}{(x - 2)^2} \\ &= (x - 2)^{-2} \\ &= f(x - 2) \\ &= f(g(x)). \end{aligned}$$

✓ ***Checkpoint*** Now try Exercise 59.

Exploration

The function in Example 10 can be decomposed in other ways. For which of the following pairs of functions is $h(x)$ equal to $f(g(x))$?

a. $g(x) = \dfrac{1}{x - 2}$ and $f(x) = x^2$

b. $g(x) = x^2$ and $f(x) = \dfrac{1}{x - 2}$

c. $g(x) = \dfrac{1}{x}$ and $f(x) = (x - 2)^2$

Example 11 Bacteria Count

The number N of bacteria in a refrigerated food is given by

$$N(T) = 20T^2 - 80T + 500, \qquad 2 \le T \le 14$$

where T is the temperature of the food in degrees Celsius. When the food is removed from refrigeration, the temperature of the food is given by

$$T(t) = 4t + 2, \qquad 0 \le t \le 3$$

where t is the time (in hours).

a. Find the composition $N(T(t))$ and interpret its meaning in context.
b. Find the number of bacteria in the food when $t = 2$ hours.
c. Find the time when the bacterial count reaches 2000.

Solution

a.
$$\begin{aligned} N(T(t)) &= 20(4t + 2)^2 - 80(4t + 2) + 500 \\ &= 20(16t^2 + 16t + 4) - 320t - 160 + 500 \\ &= 320t^2 + 320t + 80 - 320t - 160 + 500 \\ &= 320t^2 + 420 \end{aligned}$$

The composite function $N(T(t))$ represents the number of bacteria as a function of the amount of time the food has been out of refrigeration.

b. When $t = 2$, the number of bacteria is

$$\begin{aligned} N &= 320(2)^2 + 420 \\ &= 1280 + 420 \\ &= 1700. \end{aligned}$$

c. The bacterial count will reach $N = 2000$ when $320t^2 + 420 = 2000$. You can solve this equation for t algebraically as follows.

$$\begin{aligned} 320t^2 + 420 &= 2000 \\ 320t^2 &= 1580 \\ t^2 &= \frac{79}{16} \\ t &= \frac{\sqrt{79}}{4} \\ t &\approx 2.22 \text{ hours} \end{aligned}$$

So, the count will reach 2000 when $t \approx 2.22$ hours. When you solve this equation, note that the negative value is rejected because it is not in the domain of the composite function. You can use a graphing utility to confirm your solution. First graph the equation $N = 320t^2 + 420$, as shown in Figure 1.65. Then use the *zoom* and *trace* features to approximate $N = 2000$ when $t \approx 2.22$, as shown in Figure 1.66.

Checkpoint Now try Exercise 79.

Exploration

Use a graphing utility to graph $y = 320t^2 + 420$ and $y = 2000$ in the same viewing window. (Use a viewing window in which $0 \le x \le 3$ and $400 \le y \le 4000$.) Explain how the graphs can be used to answer the question asked in Example 11(c). Compare your answer with that given in part (c). When will the bacteria count reach 3200?

Notice that the model for this bacteria count situation is valid only for a span of 3 hours. Now suppose that the minimum number of bacteria in the food is reduced from 420 to 100. Will the number of bacteria still reach a level of 2000 within the three-hour time span? Will the number of bacteria reach a level of 3200 within 3 hours?

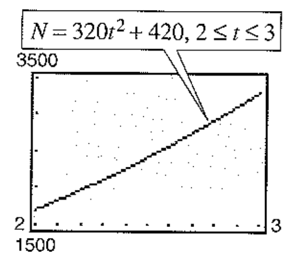


Figure 1.65

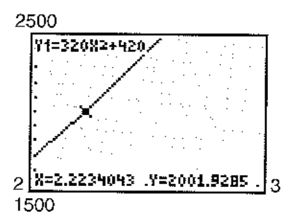


Figure 1.66

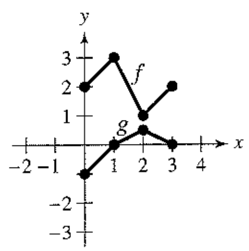
1.5 Exercises

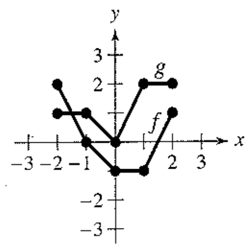
Vocabulary Check

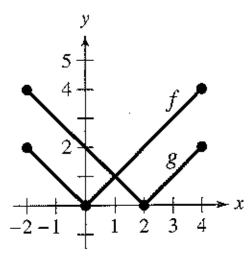
Fill in the blanks.

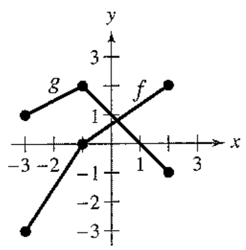
1. Two functions f and g can be combined by the arithmetic operations of ______, ______, ______, and ______ to create new functions.
2. The ______ of the function f with g is $(f \circ g)(x) = f(g(x))$.
3. The domain of $f \circ g$ is the set of all x in the domain of g such that ______ is in the domain of f.
4. To decompose a composite function, look for an ______ and ______ function.

In Exercises 1–4, use the graphs of f and g to graph $h(x) = (f + g)(x)$. To print an enlarged copy of the graph, go to the website *www.mathgraphs.com*.

1. 

2. 

3. 

4. 

In Exercises 5–12, find (a) $(f + g)(x)$, (b) $(f - g)(x)$, (c) $(fg)(x)$, and (d) $(f/g)(x)$. What is the domain of f/g?

5. $f(x) = x + 3, \quad g(x) = x - 3$
6. $f(x) = 2x - 5, \quad g(x) = 1 - x$
7. $f(x) = x^2, \quad g(x) = 1 - x$
8. $f(x) = 2x - 5, \quad g(x) = 4$
9. $f(x) = x^2 + 5, \quad g(x) = \sqrt{1 - x}$
10. $f(x) = \sqrt{x^2 - 4}, \quad g(x) = \dfrac{x^2}{x^2 + 1}$
11. $f(x) = \dfrac{1}{x}, \quad g(x) = \dfrac{1}{x^2}$
12. $f(x) = \dfrac{x}{x + 1}, \quad g(x) = x^3$

In Exercises 13–26, evaluate the indicated function for $f(x) = x^2 + 1$ and $g(x) = x - 4$ algebraically. If possible, use a graphing utility to verify your answer.

13. $(f + g)(3)$
14. $(f - g)(-2)$
15. $(f - g)(0)$
16. $(f + g)(1)$
17. $(fg)(4)$
18. $(fg)(-6)$
19. $\left(\dfrac{f}{g}\right)(-5)$
20. $\left(\dfrac{f}{g}\right)(0)$
21. $(f - g)(2t)$
22. $(f + g)(t - 4)$
23. $(fg)(-5t)$
24. $(fg)(3t^2)$
25. $\left(\dfrac{f}{g}\right)(-t)$
26. $\left(\dfrac{f}{g}\right)(t + 2)$

In Exercises 27–30, use a graphing utility to graph the functions f, g, and $f + g$ in the same viewing window.

27. $f(x) = \frac{1}{2}x, \quad g(x) = x - 1$
28. $f(x) = \frac{1}{3}x, \quad g(x) = -x + 4$
29. $f(x) = x^2, \quad g(x) = -2x$
30. $f(x) = 4 - x^2, \quad g(x) = x$

In Exercises 31–34, use a graphing utility to graph f, g, and $f + g$ in the same viewing window. Which function contributes most to the magnitude of the sum when $0 \le x \le 2$? Which function contributes most to the magnitude of the sum when $x > 6$?

31. $f(x) = 3x, \quad g(x) = -\dfrac{x^3}{10}$
32. $f(x) = \dfrac{x}{2}, \quad g(x) = \sqrt{x}$
33. $f(x) = 3x + 2, \quad g(x) = -\sqrt{x + 5}$
34. $f(x) = x^2 - \frac{1}{2}, \quad g(x) = -3x^2 - 1$

In Exercises 35–38, find (a) $f \circ g$, (b) $g \circ f$, and, if possible, (c) $(f \circ g)(0)$.

35. $f(x) = x^2, \quad g(x) = x - 1$

36. $f(x) = \sqrt[3]{x - 1}, \quad g(x) = x^3 + 1$

37. $f(x) = 3x + 5, \quad g(x) = 5 - x$

38. $f(x) = x^3, \quad g(x) = \dfrac{1}{x}$

In Exercises 39–44, (a) find $f \circ g$, $g \circ f$, and the domain of $f \circ g$. (b) Use a graphing utility to graph $f \circ g$ and $g \circ f$. Determine whether $f \circ g = g \circ f$.

39. $f(x) = \sqrt{x + 4}, \quad g(x) = x^2$

40. $f(x) = \sqrt[3]{x + 1}, \quad g(x) = x^3 - 1$

41. $f(x) = \frac{1}{3}x - 3, \quad g(x) = 3x + 1$

42. $f(x) = \sqrt{x}, \quad g(x) = \sqrt{x}$

43. $f(x) = x^{2/3}, \quad g(x) = x^6$

44. $f(x) = |x|, \quad g(x) = x + 6$

In Exercises 45–50, (a) find $(f \circ g)(x)$ and $(g \circ f)(x)$, (b) determine algebraically whether $(f \circ g)(x) = (g \circ f)(x)$, and (c) verify your answer to part (b) by comparing a table of values for each composition.

45. $f(x) = 5x + 4, \quad g(x) = 4 - x$

46. $f(x) = \frac{1}{4}(x - 1), \quad g(x) = 4x + 1$

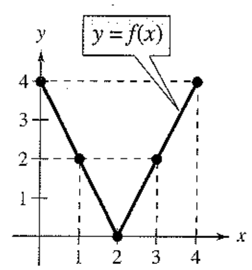
47. $f(x) = \sqrt{x + 6}, \quad g(x) = x^2 - 5$

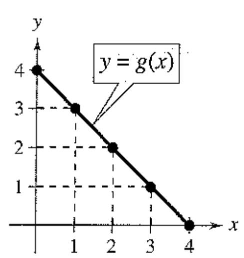
48. $f(x) = x^3 - 4, \quad g(x) = \sqrt[3]{x + 10}$

49. $f(x) = |x + 3|, \quad g(x) = 2x - 1$

50. $f(x) = \dfrac{6}{3x - 5}, \quad g(x) = -x$

In Exercises 51–54, use the graphs of f and g to evaluate the functions.





51. (a) $(f + g)(3)$ (b) $(f/g)(2)$

52. (a) $(f - g)(1)$ (b) $(fg)(4)$

53. (a) $(f \circ g)(2)$ (b) $(g \circ f)(2)$

54. (a) $(f \circ g)(1)$ (b) $(g \circ f)(3)$

In Exercises 55–62, find two functions f and g such that $(f \circ g)(x) = h(x)$. (There are many correct answers.)

55. $h(x) = (2x + 1)^2$

56. $h(x) = (1 - x)^3$

57. $h(x) = \sqrt[3]{x^2 - 4}$

58. $h(x) = \sqrt{9 - x}$

59. $h(x) = \dfrac{1}{x + 2}$

60. $h(x) = \dfrac{4}{(5x + 2)^2}$

61. $h(x) = (x + 4)^2 + 2(x + 4)$

62. $h(x) = (x + 3)^{3/2} + 4(x + 3)^{1/2}$

In Exercises 63–72, determine the domains of (a) f, (b) g, and (c) $f \circ g$. Use a graphing utility to verify your results.

63. $f(x) = \sqrt{x + 4}, \quad g(x) = x^2$

64. $f(x) = \sqrt{x + 3}, \quad g(x) = \dfrac{x}{2}$

65. $f(x) = x^2 + 1, \quad g(x) = \sqrt{x}$

66. $f(x) = x^{1/4}, \quad g(x) = x^4$

67. $f(x) = \dfrac{1}{x}, \quad g(x) = x + 3$

68. $f(x) = \dfrac{1}{x}, \quad g(x) = \dfrac{1}{2x}$

69. $f(x) = |x - 4|, \quad g(x) = 3 - x$

70. $f(x) = \dfrac{2}{|x|}, \quad g(x) = x - 1$

71. $f(x) = x + 2, \quad g(x) = \dfrac{1}{x^2 - 4}$

72. $f(x) = \dfrac{3}{x^2 - 1}, \quad g(x) = x + 1$

73. ***Stopping Distance*** The research and development department of an automobile manufacturer has determined that when required to stop quickly to avoid an accident, the distance (in feet) a car travels during the driver's reaction time is given by $R(x) = \frac{3}{4}x$, where x is the speed of the car in miles per hour. The distance (in feet) traveled while the driver is braking is given by $B(x) = \frac{1}{15}x^2$.

(a) Find the function that represents the total stopping distance T.

(b) Use a graphing utility to graph the functions R, B, and T in the same viewing window for $0 \le x \le 60$.

(c) Which function contributes most to the magnitude of the sum at higher speeds? Explain.

74. ***Sales*** From 2000 to 2005, the sales R_1 (in thousands of dollars) for one of two restaurants owned by the same parent company can be modeled by

$$R_1 = 480 - 8t - 0.8t^2, \qquad t = 0, 1, 2, 3, 4, 5$$

where $t = 0$ represents 2000. During the same six-year period, the sales R_2 (in thousands of dollars) for the second restaurant can be modeled by

$$R_2 = 254 + 0.78t, \qquad t = 0, 1, 2, 3, 4, 5.$$

(a) Write a function R_3 that represents the total sales for the two restaurants.

(b) Use a graphing utility to graph R_1, R_2, and R_3 (the total sales function) in the same viewing window.

Data Analysis **In Exercises 75 and 76, use the table, which shows the total amount spent (in billions of dollars) on health services and supplies in the United States and Puerto Rico for the years 1994 through 2000. The variables y_1, y_2, and y_3 represent out-of-pocket payments, insurance premiums, and other types of payments, respectively.** (Source: U.S. Centers for Medicare and Medicaid Services)

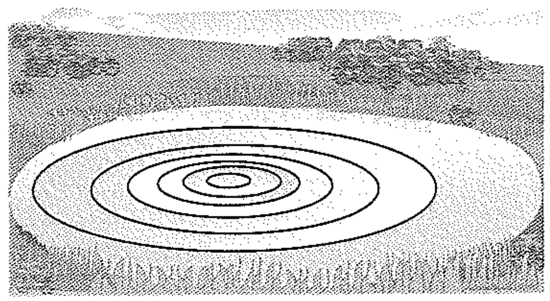
Year	y_1	y_2	y_3
1994	143.9	312.1	40.7
1995	146.5	330.1	44.9
1996	152.1	344.8	48.2
1997	162.3	359.4	52.1
1998	174.5	383.2	55.6
1999	184.4	409.4	57.3
2000	194.5	443.9	57.2

Models for the data are $y_1 = 8.93t + 103.0$, $y_2 = 1.886t^2 - 5.24t + 305.7$, and $y_3 = -0.361t^2 + 7.97t + 14.2$, where t represents the year, with $t = 4$ corresponding to 1994.

75. Use the models and the *table* feature of a graphing utility to create tables showing the values for y_1, y_2, and y_3 for each year from 1994 to 2000. Compare these values with the original data.

76. Use a graphing utility to graph y_1, y_2, y_3, and $y_1 + y_2 + y_3$ in the same viewing window. Use the model $y_1 + y_2 + y_3$ to estimate the total amount spent on health services and supplies for the years 2005 and 2010.

77. ***Ripples*** A pebble is dropped into a calm pond, causing ripples in the form of concentric circles (see figure). The radius (in feet) of the outer ripple is given by $r(t) = 0.6t$, where t is the time (in seconds) after the pebble strikes the water. The area of the circle is given by $A(r) = \pi r^2$. Find and interpret $(A \circ r)(t)$.

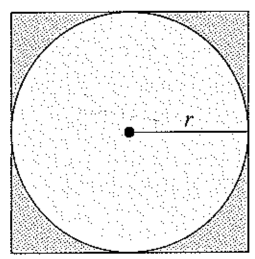


78. ***Geometry*** A square concrete foundation was prepared as a base for a large cylindrical gasoline tank (see figure).

(a) Write the radius r of the tank as a function of the length x of the sides of the square.

(b) Write the area A of the circular base of the tank as a function of the radius r.

(c) Find and interpret $(A \circ r)(x)$.



79. ***Cost*** The weekly cost C of producing x units in a manufacturing process is given by

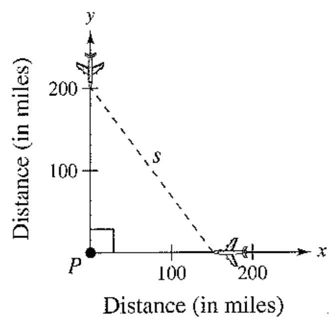
$$C(x) = 60x + 750.$$

The number of units x produced in t hours is $x(t) = 50t$.

(a) Find and interpret $(C \circ x)(t)$.

(b) Use a graphing utility to graph the cost as a function of time. Use the *trace* feature to estimate (to two-decimal-place accuracy) the time that must elapse until the cost increases to $15,000.

80. ***Air Traffic Control*** An air traffic controller spots two planes at the same altitude flying toward each other. Their flight paths form a right angle at point P. One plane is 150 miles from point P and is moving at 450 miles per hour. The other plane is 200 miles from point P and is moving at 450 miles per hour. Write the distance s between the planes as a function of time t.



81. ***Salary*** You are a sales representative for an automobile manufacturer. You are paid an annual salary plus a bonus of 3% of your sales over \$500,000. Consider the two functions

$f(x) = x - 500{,}000$ and $g(x) = 0.03x.$

If x is greater than \$500,000, which of the following represents your bonus? Explain.

(a) $f(g(x))$ (b) $g(f(x))$

82. ***Consumer Awareness*** The suggested retail price of a new car is p dollars. The dealership advertised a factory rebate of \$1200 and an 8% discount.

(a) Write a function R in terms of p giving the cost of the car after receiving the rebate from the factory.

(b) Write a function S in terms of p giving the cost of the car after receiving the dealership discount.

(c) Form the composite functions $(R \circ S)(p)$ and $(S \circ R)(p)$ and interpret each.

(d) Find $(R \circ S)(18{,}400)$ and $(S \circ R)(18{,}400)$. Which yields the lower cost for the car? Explain.

Synthesis

True or False? **In Exercises 83 and 84, determine whether the statement is true or false. Justify your answer.**

83. If $f(x) = x + 1$ and $g(x) = 6x$, then

$(f \circ g)(x) = (g \circ f)(x).$

84. If you are given two functions $f(x)$ and $g(x)$, you can calculate $(f \circ g)(x)$ if and only if the range of g is a subset of the domain of f.

85. ***Proof*** Prove that the product of two odd functions is an even function, and that the product of two even functions is an even function.

86. ***Conjecture*** Use examples to hypothesize whether the product of an odd function and an even function is even or odd. Then prove your hypothesis.

87. ***Proof*** Given a function f, prove that $g(x)$ is even and $h(x)$ is odd, where $g(x) = \frac{1}{2}[f(x) + f(-x)]$ and $h(x) = \frac{1}{2}[f(x) - f(-x)]$.

88. (a) Use the result of Exercise 87 to prove that any function can be written as a sum of even and odd functions. (*Hint:* Add the two equations in Exercise 87.)

(b) Use the result of part (a) to write each function as a sum of even and odd functions.

$$f(x) = x^2 - 2x + 1, \quad g(x) = \frac{1}{x+1}$$

Review

In Exercises 89–92, find three points that lie on the graph of the equation.

89. $y = -x^2 + x - 5$ **90.** $y = \frac{1}{5}x^3 - 4x^2 + 1$

91. $x^2 + y^2 = 24$ **92.** $y = \dfrac{x}{x^2 - 5}$

In Exercises 93–96, find an equation of the line that passes through the two points.

93. $(-4, -2), (-3, 8)$ **94.** $(1, 5), (-8, 2)$

95. $\left(\frac{3}{2}, -1\right), \left(-\frac{1}{3}, 4\right)$ **96.** $(0, 1.1), (-4, 3.1)$

In Exercises 97–102, use the graph of f to sketch the graph of the specified function. To print an enlarged copy of the graph, go to the website *www.mathgraphs.com*.

97. $f(x - 4)$

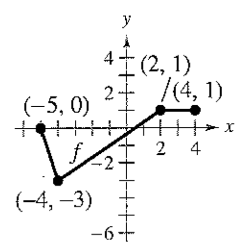
98. $f(x + 2)$

99. $f(x) + 4$

100. $f(x) - 1$

101. $2f(x)$

102. $f\left(\frac{1}{2}x\right)$



1.6 Inverse Functions

Inverse Functions

Recall from Section 1.2 that a function can be represented by a set of ordered pairs. For instance, the function $f(x) = x + 4$ from the set $A = \{1, 2, 3, 4\}$ to the set $B = \{5, 6, 7, 8\}$ can be written as follows.

$$f(x) = x + 4: \ \{(1, 5), (2, 6), (3, 7), (4, 8)\}$$

In this case, by interchanging the first and second coordinates of each of these ordered pairs, you can form the **inverse function** of f, which is denoted by f^{-1}. It is a function from the set B to the set A, and can be written as follows.

$$f^{-1}(x) = x - 4: \ \{(5, 1), (6, 2), (7, 3), (8, 4)\}$$

Note that the domain of f is equal to the range of f^{-1}, and vice versa, as shown in Figure 1.67. Also note that the functions f and f^{-1} have the effect of "undoing" each other. In other words, when you form the composition of f with f^{-1} or the composition of f^{-1} with f, you obtain the identity function.

$$f(f^{-1}(x)) = f(x - 4) = (x - 4) + 4 = x$$

$$f^{-1}(f(x)) = f^{-1}(x + 4) = (x + 4) - 4 = x$$

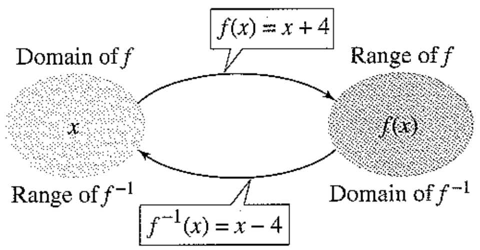


Figure 1.67

What you should learn

- Find inverse functions informally and verify that two functions are inverse functions of each other.
- Use graphs of functions to decide whether functions have inverse functions.
- Determine if functions are one-to-one.
- Find inverse functions algebraically.

Why you should learn it

Inverse functions can be helpful in further exploring how two variables relate to each other. Exercise 84 on page 71 investigates the relationship between the hourly wage and the number of units produced.

Brownie Harris/Corbis

Example 1 Finding Inverse Functions Informally

Find the inverse function of $f(x) = 4x$. Then verify that both $f(f^{-1}(x))$ and $f^{-1}(f(x))$ are equal to the identity function.

Solution

The function f *multiplies* each input by 4. To "undo" this function, you need to *divide* each input by 4. So, the inverse function of $f(x) = 4x$ is given by

$$f^{-1}(x) = \frac{x}{4}.$$

You can verify that both $f(f^{-1}(x))$ and $f^{-1}(f(x))$ are equal to the identity function as follows.

$$f(f^{-1}(x)) = f\left(\frac{x}{4}\right) = 4\left(\frac{x}{4}\right) = x \qquad f^{-1}(f(x)) = f^{-1}(4x) = \frac{4x}{4} = x$$

Checkpoint Now try Exercise 1.

STUDY TIP

Don't be confused by the use of -1 to denote the inverse function f^{-1}. In this text, whenever f^{-1} is written, it always refers to the inverse function of the function f and not to the reciprocal of $f(x)$, which is given by

$$\frac{1}{f(x)}.$$

Example 2 Finding Inverse Functions Informally

Find the inverse function of $f(x) = x - 6$. Then verify that both $f(f^{-1}(x))$ and $f^{-1}(f(x))$ are equal to the identity function.

Solution

The function f *subtracts* 6 from each input. To "undo" this function, you need to *add* 6 to each input. So, the inverse function of $f(x) = x - 6$ is given by

$$f^{-1}(x) = x + 6.$$

You can verify that both $f(f^{-1}(x))$ and $f^{-1}(f(x))$ are equal to the identity function as follows.

$$f(f^{-1}(x)) = f(x + 6) = (x + 6) - 6 = x$$

$$f^{-1}(f(x)) = f^{-1}(x - 6) = (x - 6) + 6 = x$$

✓ *Checkpoint* Now try Exercise 3.

A table of values can help you understand inverse functions. For instance, the following table shows several values of the function in Example 2. Interchange the rows of this table to obtain values of the inverse function.

x	-2	-1	0	1	2
$f(x)$	-8	-7	-6	-5	-4

⇨

x	-8	-7	-6	-5	-4
$f^{-1}(x)$	-2	-1	0	1	2

In the table at the left, each output is 6 less than the input, and in the table at the right, each output is 6 more than the input.

The formal definition of an inverse function is as follows.

Definition of Inverse Function

Let f and g be two functions such that

$f(g(x)) = x$ for every x in the domain of g

and

$g(f(x)) = x$ for every x in the domain of f.

Under these conditions, the function g is the **inverse function** of the function f. The function g is denoted by f^{-1} (read "f-inverse"). So,

$f(f^{-1}(x)) = x$ and $f^{-1}(f(x)) = x$.

The domain of f must be equal to the range of f^{-1}, and the range of f must be equal to the domain of f^{-1}.

If the function g is the inverse function of the function f, it must also be true that the function f is the inverse function of the function g. For this reason, you can say that the functions f and g are *inverse functions of each other*.

Example 3 Verifying Inverse Functions Algebraically

Show that the functions are inverse functions of each other.

$$f(x) = 2x^3 - 1 \quad \text{and} \quad g(x) = \sqrt[3]{\frac{x+1}{2}}$$

Solution

$$f(g(x)) = f\left(\sqrt[3]{\frac{x+1}{2}}\right) = 2\left(\sqrt[3]{\frac{x+1}{2}}\right)^3 - 1$$

$$= 2\left(\frac{x+1}{2}\right) - 1$$

$$= x + 1 - 1$$

$$= x$$

$$g(f(x)) = g(2x^3 - 1) = \sqrt[3]{\frac{(2x^3 - 1) + 1}{2}}$$

$$= \sqrt[3]{\frac{2x^3}{2}}$$

$$= \sqrt[3]{x^3}$$

$$= x$$

✓ *Checkpoint* Now try Exercise 15.

Example 4 Verifying Inverse Functions Algebraically

Which of the functions is the inverse function of $f(x) = \dfrac{5}{x-2}$?

$$g(x) = \frac{x-2}{5} \quad \text{or} \quad h(x) = \frac{5}{x} + 2$$

Solution

By forming the composition of f with g, you have

$$f(g(x)) = f\left(\frac{x-2}{5}\right) = \frac{5}{\left(\frac{x-2}{5}\right) - 2} = \frac{25}{x - 12} \neq x.$$

Because this composition is not equal to the identity function x, it follows that g *is not* the inverse function of f. By forming the composition of f with h, you have

$$f(h(x)) = f\left(\frac{5}{x} + 2\right) = \frac{5}{\left(\frac{5}{x} + 2\right) - 2} = \frac{5}{\left(\frac{5}{x}\right)} = x.$$

So, it appears that h is the inverse function of f. You can confirm this by showing that the composition of h with f is also equal to the identity function.

✓ *Checkpoint* Now try Exercise 19.

TECHNOLOGY TIP

Most graphing utilities can graph $y = x^{1/3}$ in two ways:

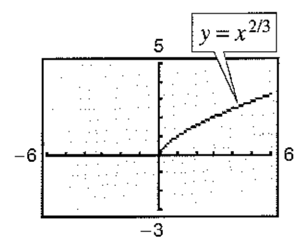
$y_1 = x \wedge (1/3)$ or

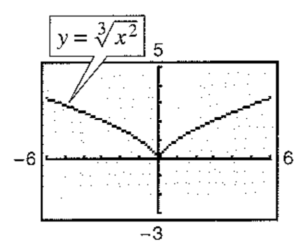
$y_1 = \sqrt[3]{x}$.

However, you may not be able to obtain the complete graph of $y = x^{2/3}$ by entering $y_1 = x \wedge (2/3)$. If not, you should use

$y_1 = (x \wedge (1/3))^2$ or

$y_1 = \sqrt[3]{x^2}$.





The Graph of an Inverse Function

The graphs of a function f and its inverse function f^{-1} are related to each other in the following way. If the point (a, b) lies on the graph of f, then the point (b, a) must lie on the graph of f^{-1}, and vice versa. This means that the graph of f^{-1} is a *reflection* of the graph of f in the line $y = x$, as shown in Figure 1.68.

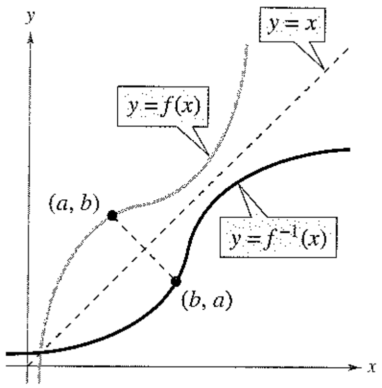


Figure 1.68

TECHNOLOGY TIP

In Examples 3 and 4, inverse functions were verified algebraically. A graphing utility can also be helpful in checking whether one function is the inverse function of another function. Use the Graph Reflection Program found on the website *college.hmco.com* to verify Example 4 graphically.

Example 5 Verifying Inverse Functions Graphically and Numerically

Verify that the functions f and g from Example 3 are inverse functions of each other graphically and numerically.

Graphical Solution

You can *graphically* verify that f and g are inverse functions of each other by using a graphing utility to graph f and g in the same viewing window. (Be sure to use a square setting.) From the graph in Figure 1.69, you can verify that the graph of g is the reflection of the graph of f in the line $y = x$.

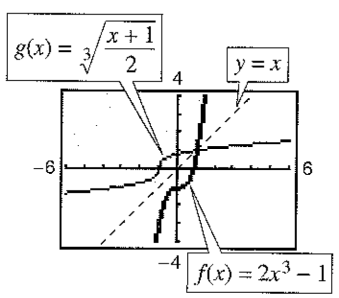


Figure 1.69

Checkpoint Now try Exercise 25.

Numerical Solution

You can *numerically* verify that f and g are inverse functions of each other. Begin by entering the compositions $f(g(x))$ and $g(f(x))$ into a graphing utility as follows.

$$y_1 = f(g(x)) = 2\left(\sqrt[3]{\frac{x+1}{2}}\right)^3 - 1$$

$$y_2 = g(f(x)) = \sqrt[3]{\frac{(2x^3 - 1) + 1}{2}}$$

Then use the *table* feature of the graphing utility to create a table, as shown in Figure 1.70. Note that the entries for x, y_1, and y_2 are the same. So, $f(g(x)) = x$ and $g(f(x)) = x$. You can now conclude that f and g are inverse functions of each other.

X	Y1	Y2
-3	-3	-3
-2	-2	-2
-1	-1	-1
0	0	0
1	1	1
2	2	2
3	3	3

X= -3

Figure 1.70

The Existence of an Inverse Function

Consider the function $f(x) = x^2$. The first table at the right is a table of values for $f(x) = x^2$. The second table was created by interchanging the rows of the first table. The second table does not represent a function because the input $x = 4$ is matched with two different outputs: $y = -2$ and $y = 2$. So, $f(x) = x^2$ does not have an inverse function.

x	-2	-1	0	1	2
$f(x)$	4	1	0	1	4

↓

x	4	1	0	1	4
$g(x)$	-2	-1	0	1	2

To have an inverse function, a function must be **one-to-one,** which means that no two elements in the domain of f correspond to the same element in the range of f.

Definition of a One-to-One Function

A function f is **one-to-one** if, for a and b in its domain, $f(a) = f(b)$ implies that $a = b$.

Existence of an Inverse Function

A function f has an inverse function f^{-1} if and only if f is one-to-one.

From its graph, it is easy to tell whether a function of x is one-to-one. Simply check to see that every horizontal line intersects the graph of the function at most once. This is called the **Horizontal Line Test.** For instance, Figure 1.71 shows the graph of $y = x^4$. On the graph, you can find a horizontal line that intersects the graph twice.

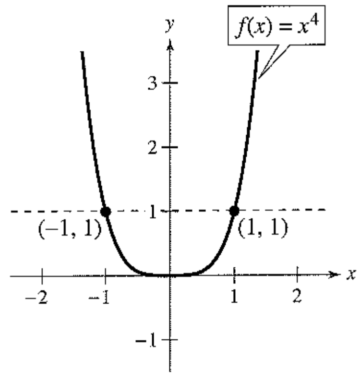


Figure 1.71 $f(x) = x^4$ is not one-to-one.

Two special types of functions that pass the Horizontal Line Test are those that are increasing or decreasing on their entire domains.

1. If f is *increasing* on its entire domain, then f is one-to-one.
2. If f is *decreasing* on its entire domain, then f is one-to-one.

Example 6 Testing for One-to-One Functions

Is the function $f(x) = \sqrt{x} + 1$ one-to-one?

Algebraic Solution

Let a and b be nonnegative real numbers with $f(a) = f(b)$.

$$\sqrt{a} + 1 = \sqrt{b} + 1 \qquad \text{Set } f(a) = f(b).$$

$$\sqrt{a} = \sqrt{b}$$

$$a = b$$

So, $f(a) = f(b)$ implies that $a = b$. We can conclude that f *is* one-to-one and *does* have an inverse function.

Graphical Solution

Use a graphing utility to graph the function $y = \sqrt{x} + 1$. From Figure 1.72, you can see that a horizontal line will intersect the graph at most once and the function is increasing. So, f *is* one-to-one and *does* have an inverse function.

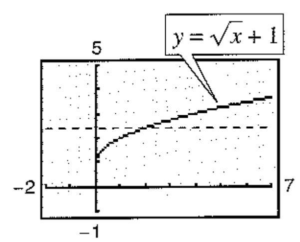


Figure 1.72

Checkpoint Now try Exercise 33.

Finding Inverse Functions Algebraically

For simple functions you can find inverse functions by inspection. For more complicated functions, however, it is best to use the following guidelines.

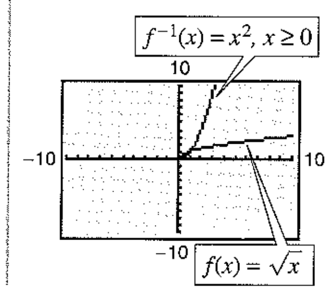
Finding an Inverse Function

1. Use the Horizontal Line Test to decide whether f has an inverse function.
2. In the equation for $f(x)$, replace $f(x)$ by y.
3. Interchange the roles of x and y, and solve for y.
4. Replace y by $f^{-1}(x)$ in the new equation.
5. Verify that f and f^{-1} are inverse functions of each other by showing that the domain of f is equal to the range of f^{-1}, the range of f is equal to the domain of f^{-1}, and $f(f^{-1}(x)) = x$ and $f^{-1}(f(x)) = x$.

It is important to note that in Step 1 above, the domain of f is assumed to be the entire real line. However, the domain of f may be restricted so that f does have an inverse function. For instance, if the domain of $f(x) = x^2$ is restricted to the nonnegative real numbers, then f does have an inverse function.

TECHNOLOGY TIP

Many graphing utilities have a built-in feature to draw an inverse function. To see how this works, consider the function $f(x) = \sqrt{x}$. The inverse function of f is given by $f^{-1}(x) = x^2$, $x \geq 0$. Enter the function $y_1 = \sqrt{x}$. Then graph it in the standard viewing window and use the *draw inverse* feature. You should obtain the figure below, which shows both f and its inverse function f^{-1}. For instructions on how to use the *draw inverse* feature, see Appendix A; for specific keystrokes, go to the text website at *college.hmco.com*.



Example 7 Finding an Inverse Function Algebraically

Find the inverse function of $f(x) = \dfrac{5 - 3x}{2}$.

Solution

The graph of f in Figure 1.73 passes the Horizontal Line Test. So you know that f is one-to-one and has an inverse function.

$$f(x) = \frac{5 - 3x}{2}$$ Write original equation.

$$y = \frac{5 - 3x}{2}$$ Replace $f(x)$ by y.

$$x = \frac{5 - 3y}{2}$$ Interchange x and y.

$$2x = 5 - 3y$$ Multiply each side by 2.

$$3y = 5 - 2x$$ Isolate the y-term.

$$y = \frac{5 - 2x}{3}$$ Solve for y.

$$f^{-1}(x) = \frac{5 - 2x}{3}$$ Replace y by $f^{-1}(x)$.

The domain and range of both f and f^{-1} consist of all real numbers. Verify that $f(f^{-1}(x)) = x$ and $f^{-1}(f(x)) = x$.

Checkpoint Now try Exercise 53.

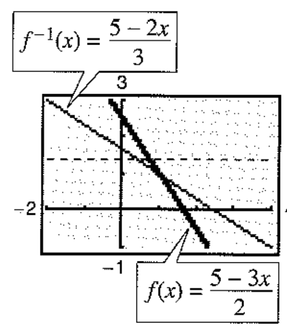


Figure 1.73

Example 8 Finding an Inverse Function Algebraically

Find the inverse function of $f(x) = x^3 - 4$ and use a graphing utility to graph f and f^{-1} in the same viewing window.

Solution

$f(x) = x^3 - 4$ Write original function.

$y = x^3 - 4$ Replace $f(x)$ by y.

$x = y^3 - 4$ Interchange x and y.

$y^3 = x + 4$ Isolate y.

$y = \sqrt[3]{x + 4}$ Solve for y.

$f^{-1}(x) = \sqrt[3]{x + 4}$ Replace y by $f^{-1}(x)$.

The graph of f in Figure 1.74 passes the Horizontal Line Test. So, you know that f is one-to-one and has an inverse function. The graph of f^{-1} in Figure 1.74 is the reflection of the graph of f in the line $y = x$.

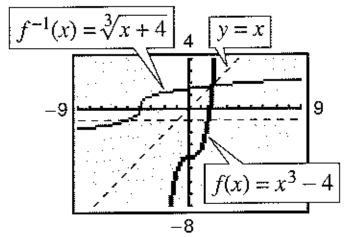


Figure 1.74

☑ ***Checkpoint*** Now try Exercise 55.

Example 9 Finding an Inverse Function Algebraically

Find the inverse function of $f(x) = \sqrt{2x - 3}$ and use a graphing utility to graph f and f^{-1} in the same viewing window.

Solution

$f(x) = \sqrt{2x - 3}$ Write original equation.

$y = \sqrt{2x - 3}$ Replace $f(x)$ by y.

$x = \sqrt{2y - 3}$ Interchange x and y.

$x^2 = 2y - 3$ Square each side.

$2y = x^2 + 3$ Isolate y.

$y = \dfrac{x^2 + 3}{2}$ Solve for y.

$f^{-1}(x) = \dfrac{x^2 + 3}{2}, \quad x \geq 0$ Replace y by $f^{-1}(x)$.

The graph of f in Figure 1.75 passes the Horizontal Line Test. So you know that f is one-to-one and has an inverse function. The graph of f^{-1} in Figure 1.75 is the reflection of the graph of f in the line $y = x$. Note that the range of f is the interval $[0, \infty)$, which implies that the domain of f^{-1} is the interval $[0, \infty)$. Moreover, the domain of f is the interval $\left[\frac{3}{2}, \infty\right)$, which implies that the range of f^{-1} is the interval $\left[\frac{3}{2}, \infty\right)$.

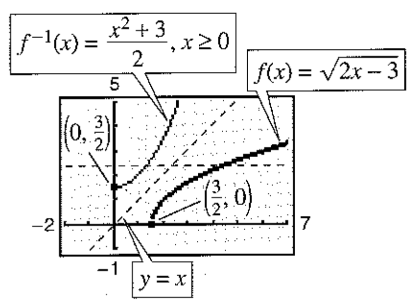


Figure 1.75

☑ ***Checkpoint*** Now try Exercise 59.

1.6 Exercises

Vocabulary Check

Fill in the blanks.

1. If the composite functions $f(g(x)) = x$ and $g(f(x)) = x$, then the function g is the _______ function of f, and is denoted by _______ .
2. The domain of f is the _______ of f^{-1}, and the _______ of f^{-1} is the range of f.
3. The graphs of f and f^{-1} are reflections of each other in the line _______ .
4. To have an inverse function, a function f must be _______ ; that is, $f(a) = f(b)$ implies $a = b$.
5. A graphical test for the existence of an inverse function is called the _______ Line Test.

In Exercises 1–8, find the inverse function of f informally. Verify that $f(f^{-1}(x)) = x$ and $f^{-1}(f(x)) = x$.

1. $f(x) = 6x$
2. $f(x) = \frac{1}{3}x$
3. $f(x) = x + 7$
4. $f(x) = x - 3$
5. $f(x) = 2x + 1$
6. $f(x) = \dfrac{x - 1}{4}$
7. $f(x) = \sqrt[3]{x}$
8. $f(x) = x^5$

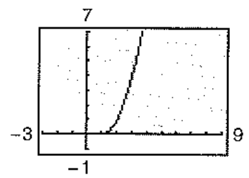
In Exercises 9–14, (a) show that f and g are inverse functions algebraically and (b) verify that f and g are inverse functions numerically by creating a table of values for each function.

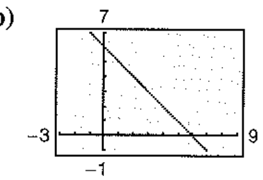
9. $f(x) = -\dfrac{7}{2}x - 3, \quad g(x) = -\dfrac{2x + 6}{7}$
10. $f(x) = \dfrac{x - 9}{4}, \quad g(x) = 4x + 9$
11. $f(x) = x^3 + 5, \quad g(x) = \sqrt[3]{x - 5}$
12. $f(x) = \dfrac{x^3}{2}, \quad g(x) = \sqrt[3]{2x}$
13. $f(x) = -\sqrt{x - 8}; \quad g(x) = 8 + x^2, \quad x \le 0$
14. $f(x) = \sqrt[3]{3x - 10}, \quad g(x) = \dfrac{x^3 + 10}{3}$

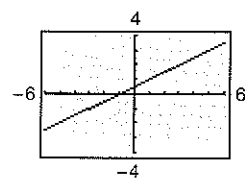
In Exercises 15–20, show that f and g are inverse functions algebraically. Use a graphing utility to graph f and g in the same viewing window. Describe the relationship between the graphs.

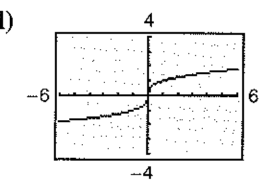
15. $f(x) = x^3, \quad g(x) = \sqrt[3]{x}$
16. $f(x) = \dfrac{1}{x}, \quad g(x) = \dfrac{1}{x}$
17. $f(x) = \sqrt{x - 4}; \quad g(x) = x^2 + 4, \; x \ge 0$
18. $f(x) = 9 - x^2, \quad x \ge 0; \quad g(x) = \sqrt{9 - x}$
19. $f(x) = 1 - x^3, \quad g(x) = \sqrt[3]{1 - x}$
20. $f(x) = \dfrac{1}{1 + x}, \quad x \ge 0; \quad g(x) = \dfrac{1 - x}{x}, \quad 0 < x \le 1$

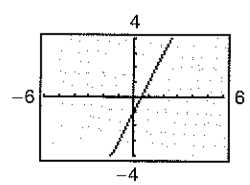
In Exercises 21–24, match the graph of the function with the graph of its inverse function. [The graphs of the inverse functions are labeled (a), (b), (c), and (d).]

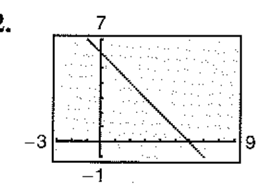
(a)


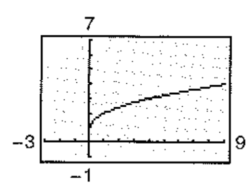
(b)


(c)


(d)


21.


22.


23.


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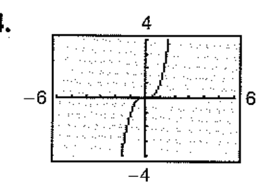


In Exercises 25–28, show that f and g are inverse functions (a) graphically and (b) numerically.

25. $f(x) = 2x, \quad g(x) = \dfrac{x}{2}$

26. $f(x) = x - 5, \quad g(x) = x + 5$

27. $f(x) = \dfrac{x - 1}{x + 5}, \quad g(x) = -\dfrac{5x + 1}{x - 1}$

28. $f(x) = \dfrac{x + 3}{x - 2}, \quad g(x) = \dfrac{2x + 3}{x - 1}$

In Exercises 29–40, use a graphing utility to graph the function and use the Horizontal Line Test to determine whether the function is one-to-one and so has an inverse function.

29. $f(x) = 3 - \frac{1}{2}x$

30. $f(x) = \frac{1}{4}(x + 2)^2 - 1$

31. $h(x) = \dfrac{x^2}{x^2 + 1}$

32. $g(x) = \dfrac{4 - x}{6x^2}$

33. $h(x) = \sqrt{16 - x^2}$

34. $f(x) = -2x\sqrt{16 - x^2}$

35. $f(x) = 10$

36. $f(x) = -0.65$

37. $g(x) = (x + 5)^3$

38. $f(x) = x^5 - 7$

39. $h(x) = |x + 4| - |x - 4|$

40. $f(x) = -\dfrac{|x - 6|}{|x + 6|}$

In Exercises 41–52, determine algebraically whether the function is one-to-one. If it is, find its inverse function. Verify your answer graphically.

41. $f(x) = x^4$

42. $g(x) = x^2 - x^4$

43. $f(x) = \dfrac{3x + 4}{5}$

44. $f(x) = 3x + 5$

45. $f(x) = \dfrac{1}{x^2}$

46. $h(x) = \dfrac{4}{x^2}$

47. $f(x) = (x + 3)^2, \quad x \ge -3$

48. $q(x) = (x - 5)^2, \quad x \le 5$

49. $f(x) = \sqrt{2x + 3}$

50. $f(x) = \sqrt{x - 2}$

51. $f(x) = |x - 2|, \quad x \le 2$

52. $f(x) = \dfrac{x^2}{x^2 + 1}$

In Exercises 53–62, find the inverse function of f. Use a graphing utility to graph both f and f^{-1} in the same viewing window. Describe the relationship between the graphs.

53. $f(x) = 2x - 3$

54. $f(x) = 3x$

55. $f(x) = x^5$

56. $f(x) = x^3 + 1$

57. $f(x) = x^{3/5}$

58. $f(x) = x^2, \quad x \ge 0$

59. $f(x) = \sqrt{4 - x^2}, \quad 0 \le x \le 2$

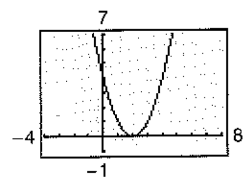
60. $f(x) = \sqrt{16 - x^2}, \quad -4 \le x \le 0$

61. $f(x) = \dfrac{4}{x}$

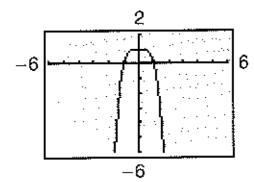
62. $f(x) = \dfrac{6}{\sqrt{x}}$

Think About It **In Exercises 63–66, delete part of the graph of the function so that the part that remains is one-to-one. Find the inverse function of the remaining part and give the domain of the inverse function. (There are many correct answers.)**

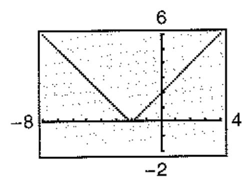
63. $f(x) = (x - 2)^2$



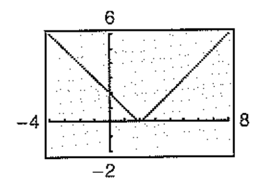
64. $f(x) = 1 - x^4$



65. $f(x) = |x + 2|$

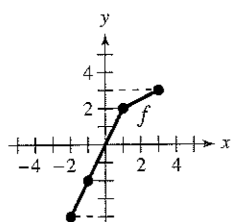


66. $f(x) = |x - 2|$



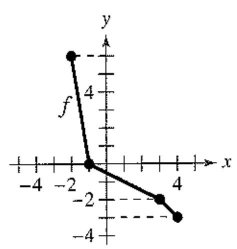
In Exercises 67 and 68, use the graph of the function f to complete the table and sketch the graph of f^{-1}.

67.



x	$f^{-1}(x)$
-4	
-2	
2	
3	

68.



x	$f^{-1}(x)$
-3	
-2	
0	
6	

Graphical Reasoning **In Exercises 69–72, (a) use a graphing utility to graph the function, (b) use the *draw inverse* feature of the graphing utility to draw the inverse of the function, and (c) determine whether the graph of the inverse relation is an inverse function, explaining your reasoning.**

69. $f(x) = x^3 + x + 1$

70. $h(x) = x\sqrt{4 - x^2}$

71. $g(x) = \dfrac{3x^2}{x^2 + 1}$

72. $f(x) = \dfrac{4x}{\sqrt{x^2 + 15}}$

In Exercises 73–78, use the functions $f(x) = \frac{1}{8}x - 3$ and $g(x) = x^3$ to find the indicated value or function.

73. $(f^{-1} \circ g^{-1})(1)$

74. $(g^{-1} \circ f^{-1})(-3)$

75. $(f^{-1} \circ f^{-1})(6)$

76. $(g^{-1} \circ g^{-1})(-4)$

77. $(f \circ g)^{-1}$

78. $g^{-1} \circ f^{-1}$

In Exercises 79–82, use the functions $f(x) = x + 4$ and $g(x) = 2x - 5$ to find the specified function.

79. $g^{-1} \circ f^{-1}$

80. $f^{-1} \circ g^{-1}$

81. $(f \circ g)^{-1}$

82. $(g \circ f)^{-1}$

83. ***Transportation*** The total value of new car sales f (in billions of dollars) in the United States from 1995 through 2001 is shown in the table. The time (in years) is given by t, with $t = 5$ corresponding to 1995. (Source: National Automobile Dealers Association)

Year, t	Sales, $f(t)$
5	456.2
6	490.0
7	507.5
8	546.3
9	606.5
10	650.3
11	690.4

(a) Does f^{-1} exist?

(b) If f^{-1} exists, what does it mean in the context of the problem?

(c) If f^{-1} exists, find $f^{-1}(650.3)$.

(d) If the table above were extended to 2002 and if the total value of new car sales for that year were \$546.3 billion, would f^{-1} exist? Explain.

84. ***Hourly Wage*** Your wage is \$8.00 per hour plus \$0.75 for each unit produced per hour. So, your hourly wage y in terms of the number of units produced is $y = 8 + 0.75x$.

(a) Find the inverse function. What does each variable in the inverse function represent?

(b) Use a graphing utility to graph the function and its inverse function.

(c) Use the *trace* feature of a graphing utility to find the hourly wage when 10 units are produced per hour.

(d) Use the *trace* feature of a graphing utility to find the number of units produced when your hourly wage is \$22.25.

Synthesis

True or False? **In Exercises 85 and 86, determine whether the statement is true or false. Justify your answer.**

85. If f is an even function, f^{-1} exists.

86. If the inverse function of f exists, and the graph of f has a y-intercept, the y-intercept of f is an x-intercept of f^{-1}.

87. ***Proof*** Prove that if f and g are one-to-one functions, $(f \circ g)^{-1}(x) = (g^{-1} \circ f^{-1})(x)$.

88. ***Proof*** Prove that if f is a one-to-one odd function, f^{-1} is an odd function.

Review

In Exercises 89–92, write the rational expression in simplest form.

89. $\dfrac{27x^3}{3x^2}$

90. $\dfrac{5x^2y}{xy + 5x}$

91. $\dfrac{x^2 - 36}{6 - x}$

92. $\dfrac{x^2 + 3x - 40}{x^2 - 3x - 10}$

In Exercises 93–98, determine whether the equation represents y as a function of x.

93. $4x - y = 3$

94. $x = 5$

95. $x^2 + y^2 = 9$

96. $x^2 + y = 8$

97. $y = \sqrt{x + 2}$

98. $x - y^2 = 0$

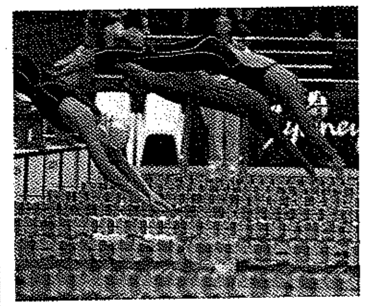
1.7 Exploring Data: Linear Models and Scatter Plots

What you should learn

- Construct scatter plots and interpret correlation.
- Use scatter plots and a graphing utility to find linear models for data.

Why you should learn it

Many real-life data follow a linear pattern. For instance, in Exercise 17 on page 79, you will find a linear model for the winning times in the women's 400-meter freestyle swimming Olympic event.



Nick Wilson/Getty Images

Scatter Plots and Correlation

Many real-life situations involve finding relationships between two variables, such as the year and the number of people in the labor force. In a typical situation, data is collected and written as a set of ordered pairs. The graph of such a set, called a *scatter plot*, is discussed briefly in Appendix B.

Example 1 Constructing a Scatter Plot

The data in the table shows the number P (in millions) of people in the United States who were part of the labor force from 1995 through 2001. Construct a scatter plot of the data. (Source: U.S. Bureau of Labor Statistics)

Year	People, P
1995	132
1996	134
1997	136
1998	138
1999	139
2000	141
2001	142

Solution

Begin by representing the data with a set of ordered pairs. Let t represent the year, with $t = 5$ corresponding to 1995.

$$(5, 132), (6, 134), (7, 136), (8, 138), (9, 139), (10, 141), (11, 142)$$

Then plot each point in a coordinate plane, as shown in Figure 1.76.

✓ ***Checkpoint*** Now try Exercise 1.

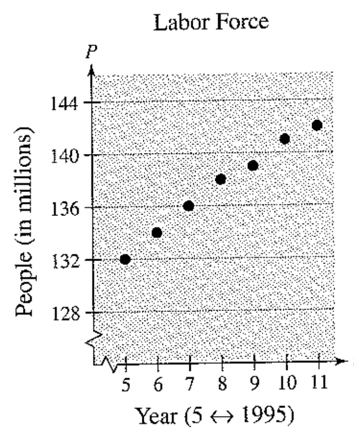
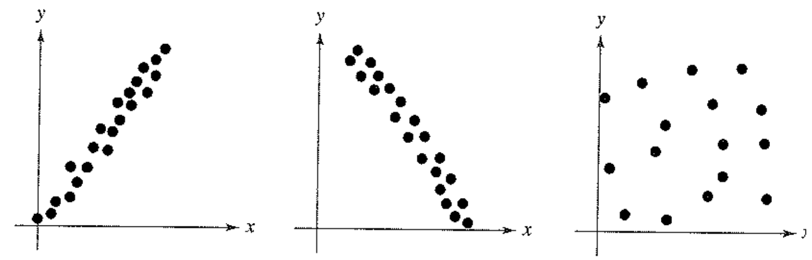


Figure 1.76

From the scatter plot in Figure 1.76, it appears that the points describe a relationship that is nearly linear. The relationship is not *exactly* linear because the labor force did not increase by precisely the same amount each year.

A mathematical equation that approximates the relationship between t and P is a *mathematical model*. When developing a mathematical model to describe a set of data, you strive for two (often conflicting) goals—accuracy and simplicity. For the data above, a linear model of the form $P = at + b$ appears to be best. It is simple and relatively accurate.

Consider a collection of ordered pairs of the form (x, y). If y tends to increase as x increases, the collection is said to have a **positive correlation.** If y tends to decrease as x increases, the collection is said to have a **negative correlation.** Figure 1.77 shows three examples: one with a positive correlation, one with a negative correlation, and one with no (discernible) correlation.



Positive Correlation Negative Correlation No Correlation
Figure 1.77

Example 2 Interpreting Correlation

On a Friday, 22 students in a class were asked to record the number of hours they spent studying for a test on Monday and the number of hours they spent watching television. The results are shown below. (The first coordinate is the number of hours and the second coordinate is the score obtained on the test.)

Study Hours: (0, 40), (1, 41), (2, 51), (3, 58), (3, 49), (4, 48), (4, 64), (5, 55), (5, 69), (5, 58), (5, 75), (6, 68), (6, 63), (6, 93), (7, 84), (7, 67), (8, 90), (8, 76), (9, 95), (9, 72), (9, 85), (10, 98)

TV Hours: (0, 98), (1, 85), (2, 72), (2, 90), (3, 67), (3, 93), (3, 95), (4, 68), (4, 84), (5, 76), (7, 75), (7, 58), (9, 63), (9, 69), (11, 55), (12, 58), (14, 64), (16, 48), (17, 51), (18, 41), (19, 49), (20, 40)

a. Construct a scatter plot for each set of data.

b. Determine whether the points are positively correlated, are negatively correlated, or have no discernable correlation. What can you conclude?

Solution

a. Scatter plots for the two sets of data are shown in Figure 1.78.

b. The scatter plot relating study hours and test scores has a positive correlation. This means that the more a student studied, the higher his or her score tended to be. The scatter plot relating television hours and test scores has a negative correlation. This means that the more time a student spent watching television, the lower his or her score tended to be.

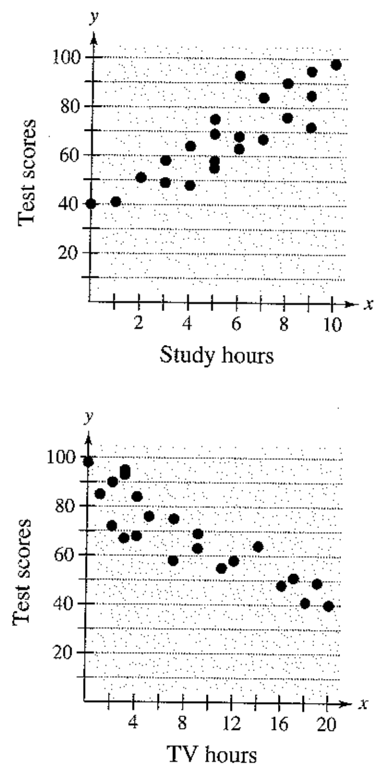


Figure 1.78

Checkpoint Now try Exercise 3.

Fitting a Line to Data

Finding a linear model to represent the relationship described by a scatter plot is called **fitting a line to data.** You can do this graphically by simply sketching the line that appears to fit the points, finding two points on the line, and then finding the equation of the line that passes through the two points.

Example 3 Fitting a Line to Data

Find a linear model that relates the year to the number of people in the United States labor force. (See Example 1.)

Year	People, P
1995	132
1996	134
1997	136
1998	138
1999	139
2000	141
2001	142

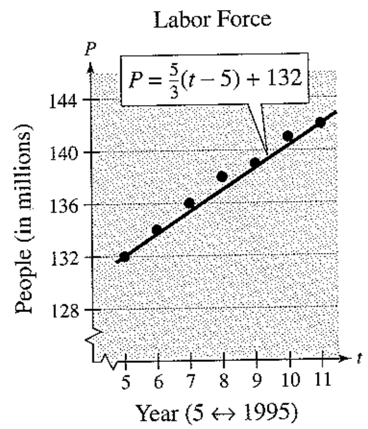


Figure 1.79

Solution

Let t represent the year, with $t = 5$ corresponding to 1995. After plotting the data in the table, draw the line that you think best represents the data, as shown in Figure 1.79. Two points that lie on this line are (5, 132) and (11, 142). Using the point-slope form, you can find the equation of the line to be

$$P = \frac{5}{3}(t - 5) + 132. \qquad \text{Linear model}$$

✓ ***Checkpoint*** Now try Exercise 11(a) and (b).

STUDY TIP

The model in Example 3 is based on the two data points chosen. If different points are chosen, the model may change somewhat. For instance, if you choose (8, 138) and (10, 141), the new model is $P = \frac{3}{2}(t - 8) + 138$.

Once you have found a model, you can measure how well the model fits the data by comparing the actual values with the values given by the model, as shown in the following table.

	t	5	6	7	8	9	10	11
Actual ⇨	P	132	134	136	138	139	141	142
Model ⇨	P	132	133.7	135.3	137	138.7	140.3	142

The sum of the squares of the differences between the actual values and the model values is the **sum of the squared differences.** The model that has the least sum is the **least squares regression line** for the data. For the model in Example 3, the sum of the squared differences is 2.16. The least squares regression line for the data is

$$P = 1.7t + 124. \qquad \text{Best-fitting linear model}$$

Its sum of squared differences is 1.04. See Appendix D for more on the least squares regression line.

Example 4 A Mathematical Model

The numbers S (in billions) of shares listed on the New York Stock Exchange for the years 1995 through 2001 are shown in the table. (Source: New York Stock Exchange, Inc.)

Year	Shares, S
1995	154.7
1996	176.9
1997	207.1
1998	239.3
1999	280.9
2000	313.9
2001	341.5

TECHNOLOGY SUPPORT

For instructions on how to use the *regression* feature, see Appendix A; for specific keystrokes, go to the text website at *college.hmco.com.*

a. Use the *regression* feature of a graphing utility to find a linear model for the data. Let t represent the year, with $t = 5$ corresponding to 1995.

b. How closely does the model represent the data?

Graphical Solution

a. Enter the data into the graphing utility's list editor. Then use the *linear regression* feature to obtain the model shown in Figure 1.80. You can approximate the model to be $S = 32.44t - 14.6$.

b. You can use a graphing utility to graph the actual data and the model in the same viewing window. From Figure 1.81, it appears that the model is a good fit for the actual data.

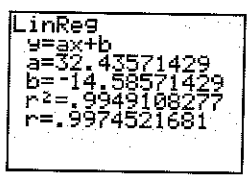


Figure 1.80

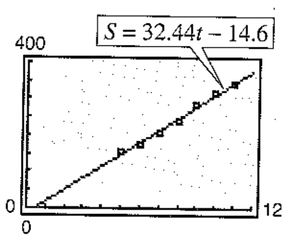


Figure 1.81

✓ *Checkpoint* Now try Exercise 15.

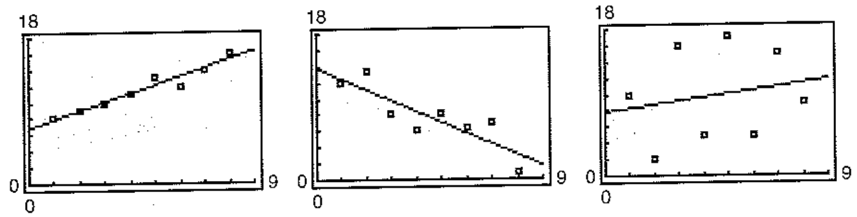
Numerical Solution

a. Using the *linear regression* feature of a graphing utility, you can find that a linear model for the data is $S = 32.44t - 14.6$.

b. You can see how well the model fits the data by comparing the actual values of S with the values of S given by the model, which are labeled S^* in the table below. From the table, you can see that the model appears to be a good fit for the actual data.

Year	S	S^*
1995	154.7	147.6
1996	176.9	180.0
1997	207.1	212.5
1998	239.3	244.9
1999	280.9	277.4
2000	313.9	309.8
2001	341.5	342.2

TECHNOLOGY TIP When you use the *regression* feature of a graphing calculator or computer program to find a linear model for data, you will notice that the program may also output an "r-value." (For some calculators, make sure you select the *diagnostic on* feature before you use the *regression* feature. Otherwise, the calculator will not output an r-value.) For instance, the r-value from Example 4 was $r \approx 0.997$. This r-value is the **correlation coefficient** of the data and gives a measure of how well the model fits the data. Correlation coefficients vary between -1 and 1. Basically, the closer $|r|$ is to 1, the better the points can be described by a line. Three examples are shown in Figure 1.82.



$r = 0.972$ $r = -0.856$ $r = 0.190$

Figure 1.82

Example 5 Finding a Least Squares Regression Line

The following ordered pairs (w, h) represent the shoe sizes w and the heights h (in inches) of 25 men. Use the *regression* feature of a graphing utility to find the least squares regression line for the data.

(10.0, 70.5)	(10.5, 71.0)	(9.5, 69.0)	(11.0, 72.0)	(12.0, 74.0)
(8.5, 67.0)	(9.0, 68.5)	(13.0, 76.0)	(10.5, 71.5)	(10.5, 70.5)
(10.0, 71.0)	(9.5, 70.0)	(10.0, 71.0)	(10.5, 71.0)	(11.0, 71.5)
(12.0, 73.5)	(12.5, 75.0)	(11.0, 72.0)	(9.0, 68.0)	(10.0, 70.0)
(13.0, 75.5)	(10.5, 72.0)	(10.5, 71.0)	(11.0, 73.0)	(8.5, 67.5)

Solution

After entering the data into a graphing utility (see Figure 1.83), you obtain the model shown in Figure 1.84. So, the least squares regression line for the data is

$$h = 1.84w + 51.9.$$

In Figure 1.85, this line is plotted with the data. Note that the plot does not have 25 points because some of the ordered pairs graph as the same point. The correlation coefficient for this model is $r \approx 0.981$, which implies that the model is a good fit for the data.

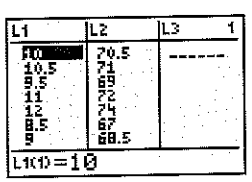


Figure 1.83

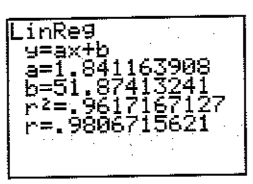


Figure 1.84

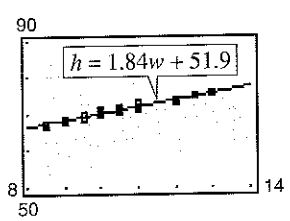


Figure 1.85

✓ ***Checkpoint*** Now try Exercise 17.

1.7 Exercises

Vocabulary Check

Fill in the blanks.

1. Consider a collection of ordered pairs of the form (x, y). If y tends to increase as x increases, then the collection is said to have a ________ correlation.
2. Consider a collection of ordered pairs of the form (x, y). If y tends to decrease as x increases, then the collection is said to have a ________ correlation.
3. The process of finding a linear model for a set of data is called ________ .
4. Correlation coefficients vary between ________ and ________ .

---

1. ***Sales*** The following ordered pairs give the years of experience x for 15 sales representatives and the monthly sales y (in thousands of dollars).

   (1.5, 41.7), (1.0, 32.4), (0.3, 19.2), (3.0, 48.4), (4.0, 51.2), (0.5, 28.5), (2.5, 50.4), (1.8, 35.5), (2.0, 36.0), (1.5, 40.0), (3.5, 50.3), (4.0, 55.2), (0.5, 29.1), (2.2, 43.2), (2.0, 41.6)

   (a) Create a scatter plot of the data.

   (b) Does the relationship between x and y appear to be approximately linear? Explain.

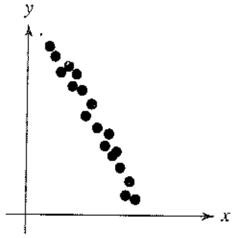
2. ***Quiz Scores*** The following ordered pairs give the scores on two consecutive 15-point quizzes for a class of 18 students.

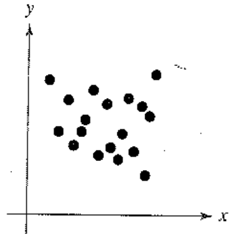
   (7, 13), (9, 7), (14, 14), (15, 15), (10, 15), (9, 7), (14, 11), (14, 15), (8, 10), (9, 10), (15, 9), (10, 11), (11, 14), (7, 14), (11, 10), (14, 11), (10, 15), (9, 6)

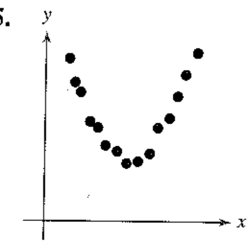
   (a) Create a scatter plot for the data.

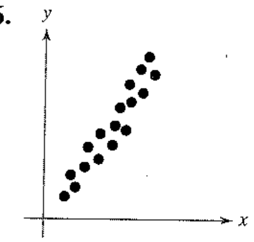
   (b) Does the relationship between consecutive quiz scores appear to be approximately linear? If not, give some possible explanations.

In Exercises 3–6, the scatter plots of sets of data are shown. Determine whether there is positive correlation, negative correlation, or no discernable correlation between the variables.

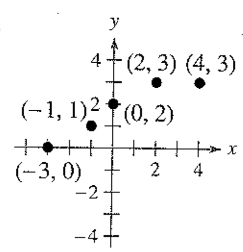
3. 

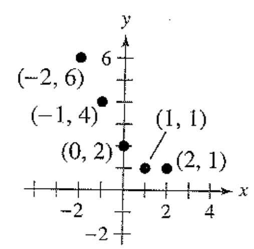
4. 

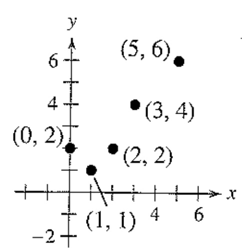
5. 

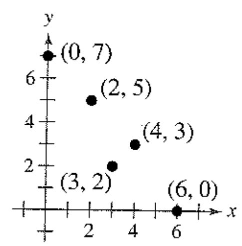
6. 

In Exercises 7–10, (a) for the data points given, draw a line of best fit through two of the points and find the equation of the line through the points, (b) use the *regression* feature of a graphing utility to find a linear model for the data, (c) graph the data points and the lines obtained in parts (a) and (b) in the same viewing window, and (d) comment on the validity of both models. To print an enlarged copy of the graph, go to the website *www.mathgraphs.com*.

7. 

8. 

9. 

10. 

11. ***Hooke's Law*** Hooke's Law states that the force F required to compress or stretch a spring (within its elastic limits) is proportional to the distance d that the spring is compressed or stretched from its original length. That is, $F = kd$, where k is the measure of the stiffness of the spring and is called the *spring constant*. The table shows the elongation d in centimeters of a spring when a force of F kilograms is applied.

Force, F	Elongation, d
20	1.4
40	2.5
60	4.0
80	5.3
100	6.6

(a) Sketch a scatter plot of the data.

(b) Find the equation of the line that seems to best fit the data.

(c) Use the *regression* feature of a graphing utility to find a linear model for the data. Compare this model with the model from part (b).

(d) Use the model from part (c) to estimate the elongation of the spring when a force of 55 kilograms is applied.

12. ***Radio*** The number R of U.S. radio stations for selected years from 1970 through 2000 is shown in the table. (Source: M Street Corporation)

Year	Radio stations, R
1970	6,760
1975	7,744
1980	8,566
1985	10,359
1990	10,788
1995	11,834
2000	13,058

(a) Use the *regression* feature of a graphing utility to find a linear model for the data. Let t represent the year, with $t = 0$ corresponding to 1970.

(b) Use a graphing utility to plot the data and graph the model in the same viewing window.

(c) Interpret the slope of the model in the context of the problem.

(d) Use the model to predict the number of radio stations in 2010.

13. ***Sports*** The average salary S (in millions of dollars) for professional baseball players from 1996 through 2002 is shown in the table. (Source: Associated Press and Major League Baseball)

Year	Salary, S
1996	1.1
1997	1.3
1998	1.4
1999	1.6
2000	1.8
2001	2.1
2002	2.3

(a) Use the *regression* feature of a graphing utility to find a linear model for the data. Let t represent the year, with $t = 6$ corresponding to 1996.

(b) Use a graphing utility to plot the data and graph the model in the same viewing window.

(c) Interpret the slope of the model in the context of the problem.

(d) Use the model to predict the average salary for a professional baseball player in 2006.

14. ***Number of Stores*** The table shows the number T of Target stores from 1997 to 2002. (Source: Target Corp.)

Year	Number of stores, T
1997	1130
1998	1182
1999	1243
2000	1307
2001	1381
2002	1476

(a) Use the *regression* feature of a graphing utility to find a linear model for the data. Let t represent the year, with $t = 7$ corresponding to 1997.

(b) Use a graphing utility to plot the data and graph the model in the same viewing window.

(c) Interpret the slope of the model in the context of the problem.

(d) Use the model to find the year in which the number of Target stores will exceed 1800.

(e) Create a table showing the actual values of T and the values of T given by the model. How closely does the model represent the data?

15. ***Communications*** The table shows the average monthly spending S (in dollars) on paging and messaging services in the United States from 1997 to 2002. (Source: The Strategis Group)

Year	Spending, S
1997	8.30
1998	8.50
1999	8.65
2000	8.80
2001	9.00
2002	9.25

(a) Use the *regression* feature of a graphing utility to find a linear model for the data. Let t represent the year, with $t = 7$ corresponding to 1997.

(b) Use a graphing utility to plot the data and graph the model in the same viewing window.

(c) Interpret the slope of the model in the context of the problem.

(d) Use the model to estimate the average monthly spending on paging and messaging services in 2008.

(e) Create a table showing the actual values of S and the values of S given by the model. How closely does the model represent the data?

16. ***Advertising and Sales*** The table shows the advertising expenditures x and sales volume y for a company for seven randomly selected months. Both are measured in thousands of dollars.

Month	Advertising expenditures, x	Sales volume, y
1	2.4	202
2	1.6	184
3	2.0	220
4	2.6	240
5	1.4	180
6	1.6	164
7	2.0	186

Table for 16

(a) Use the *regression* feature of a graphing utility to find a linear model for the data.

(b) Use a graphing utility to plot the data and graph the model in the same viewing window.

(c) Interpret the slope of the model in the context of the problem.

(d) Use the model to estimate sales for advertising expenditures of $1500.

17. ***Sports*** The following ordered pairs (x, y) represent the Olympic year x and the winning time y (in minutes) in the women's 400-meter freestyle swimming event. (Source: The New York Times Almanac 2003)

(1948, 5.30) (1976, 4.16)
(1952, 5.20) (1980, 4.15)
(1956, 4.91) (1984, 4.12)
(1960, 4.84) (1988, 4.06)
(1964, 4.72) (1992, 4.12)
(1968, 4.53) (1996, 4.12)
(1972, 4.32) (2000, 4.10)

(a) Use the *regression* feature of a graphing utility to find a linear model for the data. Let x represent the year, with $x = 0$ corresponding to 1950.

(b) What information is given by the sign of the slope of the model?

(c) Use a graphing utility to plot the data and graph the model in the same viewing window.

(d) How closely does the model fit the data?

(e) Can the model be used to estimate the winning times in the future? Explain.

18. ***Elections*** The data shows the percent x of the voting-age population that was registered to vote and the percent y that actually voted by state in 2000. (Source: U.S. Census Bureau)

AK (72.5, 65.6)	AL (73.6, 59.6)	AR (59.4, 49.4)
AZ (53.3, 46.7)	CA (52.8, 46.4)	CO (64.1, 53.6)
CT (62.5, 55.2)	D.C. (72.4, 65.6)	DE (67.9, 62.2)
FL (60.5, 51.6)	GA (61.1, 49.0)	HI (47.0, 39.7)
IA (72.2, 64.1)	ID (61.4, 53.9)	IL (66.7, 56.8)
IN (68.5, 58.5)	KS (67.7, 60.2)	KY (69.7, 54.9)
LA (75.4, 64.6)	MA (70.3, 60.1)	MD (65.6, 57.1)
ME (80.3, 69.2)	MI (69.1, 60.1)	MN (76.7, 67.8)
MO (74.3, 65.4)	MS (72.2, 59.8)	MT (70.0, 62.2)
NC (66.1, 53.2)	ND (91.1, 69.8)	NE (71.8, 58.9)
NH (69.6, 63.3)	NJ (63.2, 55.2)	NM (59.5, 51.3)
NV (52.3, 46.5)	NY (58.6, 51.0)	OH (67.0, 58.1)
OK (68.3, 58.3)	OR (68.2, 60.8)	PA (65.3, 55.7)
RI (69.7, 60.1)	SC (68.0, 58.9)	SD (70.9, 58.7)
TN (62.1, 52.3)	TX (61.4, 48.2)	UT (64.7, 56.3)
VA (64.1, 57.2)	VT (72.0, 63.3)	WA (66.1, 58.6)
WI (76.5, 67.8)	WV (63.1, 52.1)	WY (68.6, 62.5)

(a) Use the *regression* feature of a graphing utility to find a linear model for the data.

(b) Use a graphing utility to plot the data and graph the model in the same viewing window.

(c) Interpret the graph in part (b). Use the graph to identify any states that appear to differ substantially from most of the others.

(d) Interpret the slope of the model in the context of the problem.

Synthesis

True or False? **In Exercises 19 and 20, determine whether the statement is true or false. Justify your answer.**

19. A linear regression model with a positive correlation will have a slope that is greater than 0.

20. If the correlation coefficient for a linear regression model is close to -1, the regression line cannot be used to describe the data.

21. ***Writing*** A linear mathematical model for predicting prize winnings at a race is based on data for 3 years. Write a paragraph discussing the potential accuracy or inaccuracy of such a model.

22. ***Research Project*** Use your school's library, the Internet, or some other reference source to locate data that you think describes a linear relationship. Create a scatter plot of the data and find the least squares regression line that represents the points. Interpret the slope and y-intercept in the context of the data. Write a summary of your findings.

Review

In Exercises 23–26, use inequality and interval notation to describe the set.

23. P is no more than 2.

24. x is positive.

25. z is at least -3 and at most 10.

26. W is less than 7 but no less than -6.

In Exercises 27 and 28, simplify the complex fraction.

27. $\dfrac{x^2 - 4}{\left(\dfrac{x+2}{5}\right)}$ **28.** $\dfrac{\left(\dfrac{x}{x^2 + 3x - 10}\right)}{\left(\dfrac{x^2 + 3x}{x^2 + 6x + 5}\right)}$

In Exercises 29–32, evaluate the function at each value of the independent variable and simplify.

29. $f(x) = 2x^2 - 3x + 5$

(a) $f(-1)$ (b) $f(w + 2)$

30. $g(x) = 5x^2 - 6x + 1$

(a) $g(-2)$ (b) $g(z - 2)$

31. $h(x) = \begin{cases} 1 - x^2, & x \le 0 \\ 2x + 3, & x > 0 \end{cases}$

(a) $h(1)$ (b) $h(0)$

32. $k(x) = \begin{cases} 5 - 2x, & x < -1 \\ x^2 + 4, & x \ge -1 \end{cases}$

(a) $k(-3)$ (b) $k(-1)$

In Exercises 33–38, solve the equation algebraically. Check your solution graphically.

33. $6x + 1 = -9x - 8$ **34.** $3(x - 3) = 7x + 2$

35. $8x^2 - 10x - 3 = 0$ **36.** $10x^2 - 23x - 5 = 0$

37. $2x^2 - 7x + 4 = 0$ **38.** $2x^2 - 8x + 5 = 0$

1 Chapter Summary

What did you learn?

	Review Exercises
Section 1.1	
☐ Find the slopes of lines.	1–6
☐ Write linear equations given points on lines and their slopes.	7–16
☐ Use slope-intercept forms of linear equations to sketch lines.	17–24
☐ Use slope to identify parallel and perpendicular lines.	25–28
Section 1.2	
☐ Decide whether relations between two variables represent a function.	29–34
☐ Use function notation and evaluate functions.	35–38
☐ Find the domains of functions.	39–44
☐ Use functions to model and solve real-life problems.	45, 46
☐ Evaluate difference quotients.	47, 48
Section 1.3	
☐ Find the domains and ranges of functions and use the Vertical Line Test for functions.	49–56
☐ Determine intervals on which functions are increasing, decreasing, or constant.	57–60
☐ Determine relative maximum and relative minimum values of functions.	61–64
☐ Identify and graph step functions and other piecewise-defined functions.	65, 66
☐ Identify even and odd functions.	67, 68
Section 1.4	
☐ Recognize graphs of common functions.	69–72
☐ Use vertical and horizontal shifts and reflections to graph functions.	73–80
☐ Use nonrigid transformations to graph functions.	81–84
Section 1.5	
☐ Add, subtract, multiply, and divide functions.	85–90
☐ Find compositions of one function with another function.	91–94
☐ Use combinations of functions to model and solve real-life problems.	95, 96
Section 1.6	
☐ Find inverse functions informally and verify that two functions are inverse functions of each other.	97, 98
☐ Use graphs of functions to decide whether functions have inverse functions.	99, 100
☐ Determine if functions are one-to-one.	101–104
☐ Find inverse functions algebraically.	105–108
Section 1.7	
☐ Construct scatter plots and interpret correlation.	109, 110
☐ Use scatter plots and a graphing utility to find linear models for data.	111–116

1 Review Exercises

1.1 In Exercises 1–6, plot the two points and find the slope of the line passing through the pair of points.

1. $(-3, 2), (8, 2)$
2. $(7, -1), (7, 12)$
3. $(\frac{3}{2}, 1), (5, \frac{5}{2})$
4. $(-\frac{3}{4}, \frac{5}{6}), (\frac{1}{2}, -\frac{5}{2})$
5. $(-4.5, 6), (2.1, 3)$
6. $(-2.7, -6.3), (-1, -1.2)$

In Exercises 7–16, use the point on the line and the slope of the line to find the general form of the equation of the line, and find three additional points through which the line passes. (There are many correct answers.)

	Point	*Slope*
7.	$(2, -1)$	$m = \frac{1}{4}$
8.	$(-3, 5)$	$m = -\frac{3}{2}$
9.	$(0, -5)$	$m = \frac{3}{2}$
10.	$(3, 0)$	$m = -\frac{2}{3}$
11.	$(\frac{1}{5}, -5)$	$m = -1$
12.	$(0, \frac{7}{8})$	$m = -\frac{4}{5}$
13.	$(-2, 6)$	$m = 0$
14.	$(-8, 8)$	$m = 0$
15.	$(10, -6)$	m is undefined.
16.	$(5, 4)$	m is undefined.

In Exercises 17–20, find the slope-intercept form of the equation of the line that passes through the points. Use a graphing utility to graph the line.

17. $(2, -1), (4, -1)$
18. $(0, 0), (0, 10)$
19. $(-1, 0), (6, 2)$
20. $(1, 6), (4, 2)$

Rate of Change **In Exercises 21 and 22, you are given the dollar value of a product in 2005 *and* the rate at which the value of the item is expected to change during the 5 years following. Use this information to write a linear equation that gives the dollar value V of the product in terms of the year t. (Let $t = 5$ represent 2005.)**

	2005 Value	*Rate*
21.	\$12,500	\$850 increase per year
22.	\$72.95	\$5.15 decrease per year

23. ***Sales*** During the second and third quarters of the year, an e-commerce business had sales of \$160,000 and \$185,000, respectively. The growth of sales follows a linear pattern. Estimate sales during the fourth quarter.

24. ***Depreciation*** The dollar value of a VCR in 2004 is \$85, and the product will decrease in value at an expected rate of \$10.75 per year.
    (a) Write a linear equation that gives the dollar value V of the VCR in terms of the year t. (Let $t = 4$ represent 2004.)
    (b) Use a graphing utility to graph the equation found in part (a).
    (c) Use the *value* or *trace* feature of your graphing utility to estimate the dollar value of the VCR in 2008.

In Exercises 25–28, write the slope-intercept forms of the equations of the lines through the given point (a) parallel to the given line and (b) perpendicular to the given line. Verify your result with a graphing utility (use a square setting).

	Point	*Line*
25.	$(3, -2)$	$5x - 4y = 8$
26.	$(-8, 3)$	$2x + 3y = 5$
27.	$(-6, 2)$	$x = 4$
28.	$(3, -4)$	$y = 2$

1.2 In Exercises 29 and 30, which sets of ordered pairs represent functions from A to B? Explain.

29. $A = \{10, 20, 30, 40\}$ and $B = \{0, 2, 4, 6\}$
    (a) $\{(20, 4), (40, 0), (20, 6), (30, 2)\}$
    (b) $\{(10, 4), (20, 4), (30, 4), (40, 4)\}$
    (c) $\{(40, 0), (30, 2), (20, 4), (10, 6)\}$
    (d) $\{(20, 2), (10, 0), (40, 4)\}$

30. $A = \{u, v, w\}$ and $B = \{-2, -1, 0, 1, 2\}$
    (a) $\{(v, -1), (u, 2), (w, 0), (u, -2)\}$
    (b) $\{(u, -2), (v, 2), (w, 1)\}$
    (c) $\{(u, 2), (v, 2), (w, 1), (w, 1)\}$
    (d) $\{(w, -2), (v, 0), (w, 2)\}$

In Exercises 31–34, determine whether the equation represents y as a function of x.

31. $16x - y^4 = 0$
32. $2x - y - 3 = 0$
33. $y = \sqrt{1 - x}$
34. $|y| = x + 2$

In Exercises 35–38, evaluate the function at each value of the independent variable and simplify.

35. $f(x) = x^2 + 1$

(a) $f(2)$ (b) $f(-4)$
(c) $f(t^2)$ (d) $-f(x)$

36. $g(x) = x^{4/3}$

(a) $g(8)$ (b) $g(t + 1)$
(c) $g(-27)$ (d) $g(-x)$

37. $h(x) = \begin{cases} 2x + 1, & x \le -1 \\ x^2 + 2, & x > -1 \end{cases}$

(a) $h(-2)$ (b) $h(-1)$
(c) $h(0)$ (d) $h(2)$

38. $f(x) = \dfrac{3}{2x - 5}$

(a) $f(1)$ (b) $f(-2)$
(c) $f(t)$ (d) $f(10)$

In Exercises 39–44, find the domain of the function.

39. $f(x) = (x - 1)(x + 2)$
40. $f(x) = x^2 - 4x - 32$
41. $f(x) = \sqrt{25 - x^2}$
42. $f(x) = \sqrt{x^2 + 8x}$
43. $g(s) = \dfrac{5}{3s - 9}$
44. $f(x) = \dfrac{2}{3x + 4}$

45. ***Cost*** A hand tool manufacturer produces a product for which the variable cost is \$5.35 per unit and the fixed costs are \$16,000. The company sells the product for \$8.20 and can sell all that it produces.

(a) Write the total cost C as a function of x, the number of units produced.

(b) Write the profit P as a function of x.

46. ***Consumerism*** The retail sales R (in billions of dollars) of lawn care products and services in the United States from 1994 to 2001 can be approximated by the model

$$R(t) = \begin{cases} -0.67t + 11.0, & 4 \le t \le 7 \\ 0.600t^2 - 10.06t + 50.7, & 8 \le t \le 11 \end{cases}$$

where t represents the year, with $t = 4$ corresponding to 1994. Use the *table* feature of a graphing utility to approximate the retail sales of lawn care products and services for each year from 1994 to 2001. (Source: The National Gardening Association)

In Exercises 47 and 48, find the difference quotient and simplify your answer.

47. $f(x) = 2x^2 + 3x - 1,\quad \dfrac{f(x + h) - f(x)}{h},\quad h \ne 0$

48. $f(x) = x^3 - 5x^2 + x,\quad \dfrac{f(x + h) - f(x)}{h},\quad h \ne 0$

1.3 In Exercises 49–52, use a graphing utility to graph the function and estimate its domain and range. Then find the domain and range algebraically.

49. $f(x) = 3 - 2x^2$
50. $f(x) = \sqrt{2x^2 - 1}$
51. $h(x) = \sqrt{36 - x^2}$
52. $g(x) = |x + 5|$

In Exercises 53–56, (a) use a graphing utility to graph the equation and (b) use the Vertical Line Test to determine whether y is a function of x.

53. $y = \dfrac{x^2 + 3x}{6}$
54. $y = -\dfrac{2}{3}|x + 5|$
55. $3x + y^2 = 2$
56. $x^2 + y^2 = 49$

In Exercises 57–60, (a) use a graphing utility to graph the function and (b) determine the open intervals on which the function is increasing, decreasing, or constant.

57. $f(x) = x^3 - 3x$
58. $f(x) = \sqrt{x^2 - 9}$
59. $f(x) = x\sqrt{x - 6}$
60. $f(x) = \dfrac{|x + 8|}{2}$

In Exercises 61–64, use a graphing utility to approximate (to two decimal places) any relative minimum or maximum values of the function.

61. $f(x) = (x^2 - 4)^2$
62. $f(x) = x^2 - x - 1$
63. $h(x) = 4x^3 - x^4$
64. $f(x) = x^3 - 4x^2 - 1$

In Exercises 65 and 66, sketch the graph of the piecewise-defined function by hand.

65. $f(x) = \begin{cases} 3x + 5, & x < 0 \\ x - 4, & x \ge 0 \end{cases}$

66. $f(x) = \begin{cases} x^2 + 7, & x < 1 \\ x^2 - 5x + 6, & x \ge 1 \end{cases}$

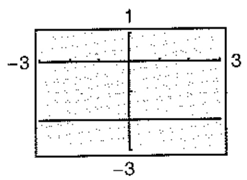
In Exercises 67 and 68, algebraically determine whether the function is even, odd, or neither. Verify your answer using a graphing utility.

67. $f(x) = (x^2 - 8)^2$

68. $f(x) = 2x^3 - x^2$

1.4 In Exercises 69–72, identify the common function and describe the transformation shown in the graph. Write an equation for the graphed function.

69.



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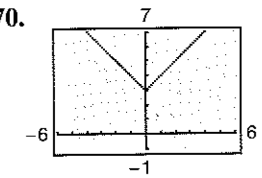

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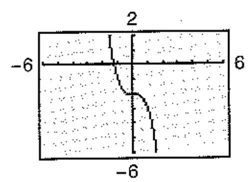

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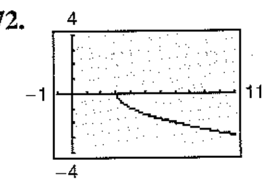

In Exercises 73–84, h is related to one of the six common functions on page 42. (a) Identify the common function f. (b) Describe the sequence of transformations from f to h. (c) Sketch the graph of h by hand. (d) Use function notation to write h in terms of the common function f.

73. $h(x) = x^2 - 6$

74. $h(x) = (x - 3)^2 - 2$

75. $h(x) = (x - 1)^3 + 7$

76. $h(x) = (x + 2)^3 + 5$

77. $h(x) = \sqrt{x} - 5$

78. $h(x) = |x + 8| - 1$

79. $h(x) = -x^2 - 3$

80. $h(x) = -(x - 2)^2 - 8$

81. $h(x) = -2x^2 + 3$

82. $h(x) = \frac{1}{2}(x - 3)^2 + 6$

83. $h(x) = -\frac{1}{2}|x| + 9$

84. $h(x) = \sqrt{3x} - 5$

1.5 In Exercises 85–94, let $f(x) = 3 - 2x$, $g(x) = \sqrt{x}$, and $h(x) = 3x^2 + 2$, and find the indicated values.

85. $(f - g)(4)$

86. $(f + h)(5)$

87. $(f + g)(25)$

88. $(g - h)(1)$

89. $(fh)(1)$

90. $\left(\frac{g}{h}\right)(1)$

91. $(h \circ g)(7)$

92. $(g \circ f)(-2)$

93. $(f \circ h)(-4)$

94. $(g \circ h)(6)$

Data Analysis **In Exercises 95 and 96, the numbers (in thousands) of students taking the SAT (y_1) and ACT (y_2) for the years 1996 through 2001 can be modeled by $y_1 = -2.75t^2 + 86.8t + 659$ and $y_2 = -1.88t^2 + 62.4t + 616$, where t represents the year, with $t = 6$ corresponding to 1996.** (Source: College Entrance Examination Board and ACT, Inc.)

95. Use a graphing utility to graph y_1, y_2, and $y_1 + y_2$ in the same viewing window.

96. Use the model $y_1 + y_2$ to estimate the total number of students taking the SAT and ACT in 2006.

1.6 In Exercises 97 and 98, find the inverse function of f informally. Verify that $f(f^{-1}(x)) = x$ and $f^{-1}(f(x)) = x$.

97. $f(x) = 6x$

98. $f(x) = x + 5$

In Exercises 99 and 100, show that f and g are inverse functions (a) graphically and (b) numerically.

99. $f(x) = 3 - 4x, \quad g(x) = \dfrac{3 - x}{4}$

100. $f(x) = \sqrt{x + 1}, \quad g(x) = x^2 - 1, x \geq 0$

In Exercises 101–104, use a graphing utility to graph the function and use the Horizontal Line Test to determine whether the function is one-to-one and so has an inverse function.

101. $f(x) = \frac{1}{2}x - 3$

102. $f(x) = (x - 1)^2$

103. $h(t) = \dfrac{2}{t - 3}$

104. $g(x) = \sqrt{x + 6}$

In Exercises 105–108, find the inverse function of f algebraically.

105. $f(x) = \dfrac{x}{12}$

106. $f(x) = \dfrac{7x + 3}{8}$

107. $f(x) = 4x^3 - 3$

108. $f(x) = \sqrt{x + 10}$

1.7

109. ***Education*** The following ordered pairs give the entrance exam scores x and the grade-point averages y after 1 year of college for 10 students.

(75, 2.3), (82, 3.0), (90, 3.6), (65, 2.0), (70, 2.1), (88, 3.5), (93, 3.9), (69, 2.0), (80, 2.8), (85, 3.3)

(a) Create a scatter plot for the data.

(b) Does the relationship between x and y appear to be approximately linear? Explain.

110. ***Stress Test*** A machine part was tested by bending it x centimeters 10 times per minute until it failed (y equals the time to failure in hours). The results are given as the following ordered pairs.

(3, 61), (6, 56), (9, 53), (12, 55), (15, 48), (18, 35), (21, 36), (24, 33), (27, 44), (30, 23)

(a) Create a scatter plot for the data.

(b) Does the relationship between x and y appear to be approximately linear? If not, give some possible explanations.

111. ***Falling Object*** In an experiment, students measured the speed s (in meters per second) of a ball t seconds after it was released. The results are shown in the table.

Time, t	Speed, s
0	0
1	11.0
2	19.4
3	29.2
4	39.4

(a) Sketch a scatter plot of the data.

(b) Find the equation of the line that seems to best fit the data.

(c) Use the *regression* feature of a graphing utility to find a linear model for the data. Compare with the model in part (b).

(d) Use the model in part (c) to estimate the speed of the ball after 2.5 seconds.

112. ***Sales*** The table shows the sales S (in millions of dollars) for Timberland from 1995 to 2002. (Source: The Timberland Co.)

Year	Sales, S
1995	655.1
1996	690.0
1997	796.5
1998	862.2
1999	917.2
2000	1091.5
2001	1183.6
2002	1190.9

(a) Use the *regression* feature of a graphing utility to find a linear model for the data. Let t represent the year, with $t = 5$ corresponding to 1995.

(b) Use a graphing utility to plot the data and graph the model in the same viewing window.

(c) Interpret the slope of the model in the context of the problem.

(d) Use the model to find the year in which the sales will exceed $1300 million.

(e) Create a table showing the actual values of S and the values of S given by the model. How closely does the model represent the data?

Height **In Exercises 113–116, the following ordered pairs (x, y) represent the percent y of women between the ages of 20 and 29 who are under a certain height x (in feet).** (Source: U.S. National Center for Health Statistics)

(4.67, 0.6)	**(5.17, 21.8)**	**(5.67, 92.4)**
(4.75, 0.7)	**(5.25, 34.3)**	**(5.75, 96.2)**
(4.83, 1.2)	**(5.33, 48.9)**	**(5.83, 98.6)**
(4.92, 3.1)	**(5.42, 62.7)**	**(5.92, 99.5)**
(5.00, 6.0)	**(5.50, 74.0)**	**(6.00, 100.0)**
(5.08, 11.5)	**(5.58, 84.7)**	

113. Use the *regression* feature of a graphing utility to find a linear model for the data.

114. Use a graphing utility to plot the data and graph the model in the same viewing window.

115. How closely does the model fit the data?

116. Can the model be used to estimate the percent of women who are under a height of greater than 6 feet?

Synthesis

True or False? **In Exercises 117–120, determine whether the statement is true or false. Justify your answer.**

117. If the graph of the common function $f(x) = x^2$ is moved six units to the right, moved three units upward, and reflected in the x-axis, then the point $(-1, 28)$ will lie on the graph of the transformation.

118. If $f(x) = x^n$ where n is odd, f^{-1} exists.

119. There exists no function f such that $f = f^{-1}$.

120. The sign of the slope of a regression line is always positive.

1 Chapter Test

Take this test as you would take a test in class. After you are finished, check your work against the answers given in the back of the book.

1. A line with slope $m = \frac{3}{2}$ passes through the point $(3, -1)$. List three additional points on the line. Then sketch the line.
2. Find an equation of the line that passes through the point $(0, 4)$ and is (a) parallel to and (b) perpendicular to the line $5x + 2y = 3$.
3. Does the graph at the right represent y as a function of x? Explain.
4. Evaluate $f(x) = |x + 2| - 15$ at each value of the independent variable and simplify.

   (a) $f(-8)$ (b) $f(14)$ (c) $f(t - 6)$
5. Find the domain of $f(x) = 10 - \sqrt{3 - x}$.
6. An electronics company produces a car stereo for which the variable cost is \$5.60 and the fixed costs are \$24,000. The product sells for \$99.50. Write the total cost C as a function of x. Write the profit P as a function of x.

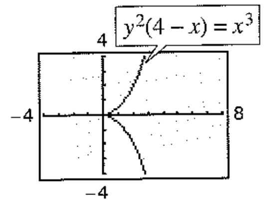

Figure for 3

In Exercises 7 and 8, determine the open intervals on which the function is increasing, decreasing, or constant.

7. $h(x) = \frac{1}{4}x^4 - 2x^2$ 8. $g(t) = |t + 2| - |t - 2|$

In Exercises 9 and 10, use a graphing utility to approximate (to two decimal places) any relative minimum or maximum values of the function.

9. $f(x) = -x^3 - 5x^2 + 12$ 10. $f(x) = x^5 - x^3 + 2$

In Exercises 11–13, (a) identify the common function f, (b) describe the sequence of transformations from f to g, and (c) sketch the graph of g.

11. $g(x) = -2(x - 5)^3 + 3$ 12. $g(x) = \sqrt{-x - 7}$ 13. $g(x) = 4|-x| - 7$
14. Use the functions $f(x) = x^2$ and $g(x) = \sqrt{2 - x}$ to find the specified function and its domain.

    (a) $(f - g)(x)$ (b) $(f/g)(x)$ (c) $(f \circ g)(x)$ (d) $(g \circ f)(x)$

In Exercises 15–17, determine whether the function has an inverse function, and if so, find the inverse function.

15. $f(x) = x^3 + 8$ 16. $f(x) = x^2 + 6$ 17. $f(x) = \dfrac{3x\sqrt{x}}{8}$
18. The table shows the number of local telephone access lines L (in millions) in the United States from 1994 through 2000, where t represents the year, with $t = 4$ corresponding to 1994. Use the *regression* feature of a graphing utility to find a linear model for the data. Use the model to find the year in which the number of local telephone access lines will exceed 300 million. (Source: U.S. Federal Communications Commission)

Year, t	Lines, L
4	157
5	166
6	178
7	194
8	205
9	228
10	245

Table for 18